Revised Version

The Based Ring of Two-Sided Cells of Affine Weyl Groups of Type $\tilde{A}_{n-1}$

Nanhua Xi

Author address:

Institute of Mathematics, Chinese Academy of Sciences, Beijing 100080, China

E-mail address: nanhua@math08.math.ac.cn

To Guiju LIU

1991 *Mathematics Subject Classification.* Primary 20G05, 18F25;
Secondary 16S80, 20C07.

The author was supported in part by Chinese National Sciences Foundation.

ABSTRACT. In this paper we prove Lusztig's conjecture on based ring for an affine Weyl group of type $\tilde{A}_{n-1}$.

Contents

Introduction

The Kazhdan-Lusztig theory deeply increases our understanding of Coxeter groups and their representations and their role in Lie representation theory. The central concepts in Kazhdan-Lusztig theory include Kazhdan-Lusztig polynomial, Kazhdan-Lusztig basis, cell, based ring. Kazhdan-Lusztig polynomials play an essential role in understanding certain remarkable representations in Lie theory, for instances, the representations of quantum groups at roots of 1, the modular representations of algebraic groups, the representations of Kac-Moody algebras. Kazhdan-Lusztig basis and cells are very useful in understanding structure and representations of Coxeter groups and their Hecke algebras.

The based ring of a two-sided cell of certain Coxeter groups is defined through Kazhdan-Lusztig basis by Lusztig in [L5]. The based ring is very interesting in understanding the concerned Hecke algebras and their representations (see [L5, L6, L8]). Moreover we can construct irreducible representations of the Hecke algebras if we know the structure of the based ring explicitly (see [X3]). This fact is remarkable since constructing computable irreducible modules of affine Hecke algebras is in general difficult. Recently Lusztig introduced periodic W-graphs for constructing finite dimensional modules of affine Hecke algebras (see [L9]).

For the structure of the based ring of a two-sided cell in a Weyl group or in an affine Weyl group, Lusztig has two nice conjectures, one for Weyl groups (see [L7]) and the other for affine Weyl groups that says the based ring is isomorphic to a certain equivariant K-group (see [L8]). The conjecture for Weyl groups is proved by Lusztig (unpublished). The conjecture for affine Weyl groups is proved for rank 2 cases, for the lowest two-sided cell of an affine Weyl group and for the second highest two-sided cell of an affine Weyl group, see [X1, X3]. As an application, the author gives a classification of irreducible modules of the Hecke algebras of affine Weyl groups of rank 2 for any non-zero parameter, see [X3]. (When the parameter is not a root of 1, Kazhdan and Lusztig worked out the classification of irreducible modules of an affine Hecke algebra, see [KL2].) Also the author computed the irreducible modules of affine Hecke algebras associated with second highest two-sided cell, see [X3]. Recently Ram has developed an interesting combinatorical approach to study representations of affine Hecke algebras, see [R1, R2].

In this paper we will prove Lusztig Conjecture on based ring for an extended affine Weyl group associated with the general linear group $GL_n(\mathbb{C})$ or the special linear group $SL_n(\mathbb{C})$, which is of type $\tilde{A}_{n-1}$.

Let us here briefly explain the idea of the proof of the conjecture for type $\tilde{A}_{n-1}$. For each two-sided cell of the extended affine Weyl group associated with $GL_n(\mathbb{C})$, we first show that the based ring of the two-sided cell is a matrix algebra over the based ring of the intersection of a left cell in the two-sided cell and its inverse (a

right cell). Then we show that the based ring of the intersection is isomorphic to the (rational) representation ring of a certain connected reductive algebraic group over complex numbers. The main difficulty is to establish a bijection between the intersection (of a left cell in the two-sided cell and its inverse) and the set of isomorphism classes of irreducible rational representations of the algebraic group, and to show that the bijection leads to the isomorphism between the based ring of the intersection and the representation ring of the algebraic group. The bijection is established in Chapter 5. In Chapters 6 and 7 we prove several formulas in the based ring of the intersection, the corresponding formulas in the representation ring of the concerned algebraic group are obvious. Using the formulas established in Chapters 6 and 7, in Chapter 8 we show that the bijection in Chapter 5 is the right one. In Chapter 8 we also show that Lusztig Conjecture on based ring is true for the extended affine Weyl group associated with $SL_n(\mathbb{C})$ and can not be generalized to arbitrary extended affine Weyl groups.

The contents of the paper are as follows.

In Chapter 1 we recall some basic definitions and facts about cells and based rings. In section 1.4 we prove some simple properties about the structure constants of a based ring that are important to our proof of Lusztig Conjecture on based ring for an affine Weyl group of type $\tilde{A}_{n-1}$. In Chapter 2 we discuss the structure of cells in the extended affine Weyl group associated with $GL_n(\mathbb{C})$. The cells in the extended affine Weyl group are explicitly described by Shi and Lusztig (see [S, L3]). In section 2.3, we show that the based ring of a two-sided cell of the extended affine Weyl group is isomorphic to a matrix algebra over the based ring of the intersection of any given left cell in the two-sided cell and its inverse (a right cell). Thus to understand the based ring of the two-sided cell, we only need to work out the structure of the based ring of the intersection of a left cell in the two-sided cell and its inverse. In section 2.4 we discuss chains and antichains defined by Shi, which are essential for our bijection between the intersection of a left cell and its inverse and the set of isomorphism classes of irreducible rational representations of the corresponding algebraic group.

In Chapter 3 we give some discussion to canonical left cells. Although we do not need the results in this chapter for the proof of our main result of the paper, but for other types canonical left cells maybe play a big role for some questions such as Lusztig Conjecture on based ring and properties of Lusztig bijection between the set of two-sided cells of an affine Weyl group and the set of unipotent classes of the corresponding algebraic group.

In Chapter 4 we describe the reductive algebraic group corresponding to a two-sided cell in the extended affine Weyl group associated with $GL_n(\mathbb{C})$ and recall some needed results about the representation ring of a general linear group over complex numbers.

In Chapter 5 we establish a bijection between the intersection of a left cell and its inverse and the set of dominant weights of the corresponding reductive algebraic group. We obtain this bijection by means of antichains and this bijection is one key to our main result.

In Chapters 6 and 7 we work with the based ring of the intersection of a left cell and its inverse. Believing in that Lusztig Conjecture is true, motivated by some simple multiplication formulas in the representation ring of the concerned reductive algebraic group, we prove some multiplication formulas in the based ring. Using

the formulas in Chapters 6 and 7, in Chapter 8 we prove that the based ring of the intersection is isomorphic to the representation ring of the corresponding reductive algebraic group. This completes our proof of Lusztig Conjecture on based ring for type $\tilde{A}_{n-1}$. In Chapter 8 we also discuss the based rings of extended affine Weyl groups associated with the projective linear group $PGL_n(\mathbb{C})$ and with the special linear group $SL_n(\mathbb{C})$.

It is expected that the explicit knowledge on the based rings will have nice application to representation theory of the concerned affine Hecke algebras. It is interesting that we can use the explicit structure on based ring to find leading coefficients of some Kazhdan-Lusztig polynomials, the details will appear elsewhere.

I am very grateful to Professors V. Chari and J.-Y. Shi for helpful discussions. I thank Professor Le Yang for his constant encouragement. I would like to thank Professor G. Lusztig for many helpful comments. I am greatly indebted to the referee for carefully reading, very valuable comments and for pointing out a serious gap in an earlier version of the paper. Part of the work was done during my visit to Department of Mathematics at University of California at Riverside, I thank the Department of Mathematics for hospitality. The author was in part supported by Chinese National Sciences Foundation.

CHAPTER 1

Cells in Affine Weyl Groups

In this chapter we first recall some basic concepts such as cell, based ring, etc., then in section 1.4 we give some discussions to star operations. In section 1.1 we recall the definition of Kazhdan-Lusztig basis of a Hecke algebra. In section 1.2 we recall the definition of cell and of the a-function. In section 1.3 we recall some properties about the integers $\gamma_{w,u,v}$ (defined through the structure constants of Kazhdan-Lusztig basis), which are due to Lusztig.

The star operation was introduced in [KL1] and is a useful tool to study cells of Coxeter groups. In section 1.4 we prove some interesting results about the relations between star operations and the integers $\gamma_{w,u,v}$. The main result in this section is Theorem 1.4.5 which says that the integers $\gamma_{w,u,v}$ are invariant under star operations. In section 1.5 we recall the definition of based ring (due to Lusztig, see [L5]) and Lusztig's nice conjecture about the structure of the based ring of a two-sided cell of an affine Weyl group. A few comments about the conjecture are given . In section 1.6 we discuss the relationship between the generalized star operations (see [L4]) and the integers $\gamma_{w,u,v}$. The results in section 1.6 should be helpful for understanding the structure of the based ring of a two-sided cell of an affine Weyl group of type $\tilde{B}_n,\ \tilde{C}_n,\ \tilde{F}_4$.

The basic references for this chapter are [KL1] and [L4-L8].

1.1. Hecke algebra

Here Hecke algebras are defined for extended Coxeter groups.

Let (W',S) be a Coxeter system with S the set of simple reflections. Assume that a commutative group Ω acts on (W',S). Then we can consider the extended Coxeter group $W=\Omega\ltimes W'$. The **length function** l on W' and the **partial order** $\le$ on W' are extended to W as usual, that is, $l(\omega w)=l(w)$, and $\omega w\le\omega' u$ if and only if $\omega=\omega'$ and $w\le u$, where ω,ω' are in Ω and w,u are in W'.

Let q be an indeterminate. The **Hecke algebra** H of (W,S) over $\mathcal{A}=\mathbb{Z}[q,q^{-1}]$ with parameter q^2 is an associative algebra over $\mathcal{A}$, with an $\mathcal{A}$-basis $\{T_w\mid w\in W\}$ and relations (1) $(T_s-q^2)(T_s+1)=0$ if $s\in S$, (2) $T_wT_u=T_{wu}$ if $l(wu)=l(w)+l(u)$.

Let $a\to\bar{a}$ be the involution of $\mathcal{A}$ defined by $\bar{q}=q^{-1}$. Then we have a bar involution of H defined by

$$\overline{\sum a_wT_w}=\sum\bar{a}_wT_{w^{-1}}^{-1},\qquad a_w\in\mathcal{A}.$$

For each $w\in W$ there is a unique element C_w in H such that $\bar{C}_w=C_w$ and $C_w=q^{-l(w)}\sum_{y\le w}P_{y,w}(q^2)T_y$, where $P_{y,w}$ is a polynomial in q of degree $\le\frac{1}{2}(l(w)-l(y)-1)$ if $l(w)>l(y)$ and $P_{w,w}=1$.

The basis $\{C_w \mid w \in W\}$ is the famous **Kazhdan-Lusztig basis** of the Hecke algebra H and the polynomials $P_{y,w}$ are the celebrated **Kazhdan-Lusztig polynomials**. The basis and the polynomials have deep relations with the geometry of Schubert varieties when W is the Coxeter group associated with a Kac-Moody algebra and the polynomials are essential in understanding some difficult irreducible representations (such as the finite dimensional irreducible representations of quantum groups at roots of 1, some irreducible representations of Kac-Moody algebras, irreducible rational representations of algebraic groups over an algebraic closed field of positive characteristic, etc.).

The basis also plays an important role in understanding the representations of the Hecke algebra H, see [KL1, L4-L8]. Through the basis Lusztig defined based ring for some Coxeter groups (including Weyl groups and affine Weyl groups). In this paper we will determine the structure of the based ring of an affine Weyl group of type $\tilde{A}_{n-1}$. Applications to the representation theory of the concerned Hecke algebras will appear elsewhere.

If $y \leq w$ and $y \neq w$, we have $P_{y,w} = \mu(y,w)q^{\frac{1}{2}(l(w)-l(y)-1)}$+lower degree terms. We write $y \prec w$ if $\mu(y,w) \neq 0$. We then set $\mu(y,w) = \mu(w,y)$. We shall write $y - w$ if $\mu(y,w) \neq 0$ or $\mu(w,y) \neq 0$. We have

(a) Assume that $y \leq w$ and $s \in S$. If $sw \leq w$, then $sy \leq w$ and $P_{y,w} = P_{sy,w}$. If $ws \leq w$, then $ys \leq w$ and $P_{y,w} = P_{ys,w}$. See [KL1].

1.2. Cell and a-function

Cells of Coxeter groups are defined in [KL1] and are useful in understanding the structure and representations of W and of the Hecke algebra H. Also we can systematically construct representations of Hecke algebras by means of the cells (see [KL1, L4, L6, L8]).

Let us recall the definition of cells of Coxeter groups. For $w \in W$ set

$$L(w) = \{s \in S \mid sw \leq w\}$$

$$R(w) = \{s \in S \mid ws \leq w\}.$$

Let w and u be elements in W'. We say that $w \underset{L}{\leq} u$ (resp. $w \underset{R}{\leq} u; w \underset{LR}{\leq} u$) if there exists a sequence $w = w_0, w_1, w_2, ..., w_k = u$ in W' such that for $i = 1, 2, ..., k$ we have $\mu(w_{i-1}, w_i) \neq 0$ and $L(w_{i-1}) \not\subseteq L(w_i)$ (resp. $R(w_{i-1}) \not\subseteq R(w_i)$; $L(w_{i-1}) \not\subseteq L(w_i)$ or $R(w_{i-1}) \not\subseteq R(w_i)$). Then for any ω, ω' in Ω we say that $\omega w \underset{L}{\leq} \omega' u$ (resp. $w\omega \underset{R}{\leq} u\omega'$; $\omega w \underset{LR}{\leq} \omega' u$) if $w \underset{L}{\leq} u$ (resp. $w \underset{R}{\leq} u$, $w \underset{LR}{\leq} u$).

For elements w, u in W we write that $w \underset{L}{\sim} u$ (resp. $w \underset{R}{\sim} u$; $w \underset{LR}{\sim} u$) if $w \underset{L}{\leq} u \underset{L}{\leq} w$ (resp. $w \underset{R}{\leq} u \underset{R}{\leq} w$; $w \underset{LR}{\leq} u \underset{LR}{\leq} w$). The relations $\underset{L}{\leq}, \underset{R}{\leq}, \underset{LR}{\leq}$ are preorders on W and the relations $\underset{L}{\sim}, \underset{R}{\sim}, \underset{LR}{\sim}$ are equivalence relations on W. The corresponding equivalence classes are called **left cells, right cells, two-sided cells** of W, respectively. The preorder $\underset{L}{\leq}$ (resp. $\underset{R}{\leq}; \underset{LR}{\leq}$) induces a partial order on the set of left (resp. right; two-sided) cells of W, denoted again by $\underset{L}{\leq}$ (resp. $\underset{R}{\leq}; \underset{LR}{\leq}$). The following properties (see [KL1] or [L4, L5]) will be needed.

(a) If $w \underset{L}{\leq} u$, then $R(w) \supseteq R(u)$. In particular, if $w \underset{L}{\sim} u$, then $R(w) = R(u)$.
(b) If $w \underset{R}{\leq} u$, then $L(w) \supseteq L(u)$. In particular, if $w \underset{R}{\sim} u$, then $L(w) = L(u)$.

For any $\xi, \xi' \in H$ and $w \in W$ we write

$$\xi C_w = \sum_{v\in W} h_v C_v, \qquad h_v \in \mathcal{A},$$

$$C_w \xi' = \sum_{v\in W} h'_v C_v, \qquad h'_v \in \mathcal{A},$$

$$\xi C_w \xi' = \sum_{v\in W} h''_v C_v, \qquad h''_v \in \mathcal{A}.$$

Then

(c) $v \underset{L}{\leq} w$ (resp. $v \underset{R}{\leq} w$; $v \underset{LR}{\leq} w$) if $h_v \neq 0$ (resp. $h'_v \neq 0$; $h''_v \neq 0$).
(d) Assume that (W', S) is crystallographic and that there exists a positive integer a_0 such that $a(v) \leq a_0$ for all $v \in W$ (see below for the definition of $a(v)$). If $h_v \neq 0$ (resp. $h'_v \neq 0$) and $w \underset{L}{\not\sim} v$ (resp. $w \underset{R}{\not\sim} v$), then $w \underset{LR}{\not\sim} v$.

The a-**function** on W introduced by Lusztig is a useful tool to study cells of W and is also necessary for defining based ring of two-sided cells. Given w, u in W we write

$$C_w C_u = \sum_{v\in W} h_{w,u,v} C_v, \qquad h_{w,u,v} \in \mathcal{A}.$$

For $v \in W$ we define $a(v)$=the minimal non-negative integer i such that $q^{-i} h_{w,u,v}$ is in $\mathbb{Z}[q^{-1}]$ for all w, u in W. If such i does not exist we set $a(v) = \infty$.

Set $\tilde{T}_w = q^{-l(w)} T_w$ for $w \in W$. Then $\tilde{T}_w \in C_w + q^{-1} \sum_{x\in W} \mathbb{Z}[q^{-1}] C_x$. Write

$$\tilde{T}_w \tilde{T}_u = \sum_{v\in W} h'_{w,u,v} C_v, \qquad h'_{w,u,v} \in \mathcal{A}.$$

Then we also have $a(v)$=the minimal non-negative integer i such that $q^{-i} h'_{w,u,v}$ is in $\mathbb{Z}[q^{-1}]$ for all w, u in W. See [L4].

1.3. Affine Weyl group

We are mainly interested in extended affine Weyl groups and their Hecke algebras (especially of type $\tilde{A}_{n-1}$). In this section we recall some properties about the integers $\gamma_{w,u,v}$ defined through the Kazhdan-Lusztig basis of an affine Hecke algebra. The integers are actually structure constants of the based ring of the extended affine Weyl group.

Let G be a connected reductive group over $\mathbb{C}$. Let W_0, X, P, R be the corresponding Weyl group, weight lattice, root lattice, root system, respectively. Then $W' = W_0 \ltimes P$ is an **affine Weyl group** and $W = W_0 \ltimes X$ is an **extended affine Weyl group**. Let S be a set of simple reflections of W' such that $S \cap W_0$ generates W_0 and is a set of simple reflections of W_0. We can find an abelian subgroup Ω of W such that $\omega S = S\omega$ for any $\omega \in \Omega$ and $W = \Omega \ltimes W'$.

Lusztig proved that the number of left cells in W is finite. Moreover each left cell of W contains a unique element of

$$\mathcal{D} = \{w \in W' \mid 2\deg P_{e,w} = l(w) - a(w)\},$$

where e is the neutral element of W. The set $\mathcal{D}$ is a finite set of involutions of W'. The elements in $\mathcal{D}$ will be called **distinguished involutions**. See [L5].

Let w_0 be the longest element of W_0. We have $a(w) \leq l(w_0) = a(w_0)$ for all w in W (see [L4]). Thus for any $w, u, v \in W$ we can define an integer $\gamma_{w,u,v}$ by the condition

$$q^{-a(v)} h_{w,u,v} - \gamma_{w,u,v} \in q^{-1}\mathbb{Z}[q^{-1}],$$

see 1.2 for the definition of $h_{w,u,v}$. The following are some properties of $\gamma_{w,u,v}$ (see [L5] for (a)-(e) and [L4] for (f)).

(a) If $\gamma_{w,u,v}$ is not equal to 0, then $w \underset{L}{\sim} u^{-1}$, $u \underset{L}{\sim} v$ and $w \underset{R}{\sim} v$. In particular we have $w \underset{LR}{\sim} u \underset{LR}{\sim} v$ if $\gamma_{w,u,v}$ is not equal to 0.

(b) $\gamma_{w,u,v} = \gamma_{u,v^{-1},w^{-1}}$ and $\gamma_{u^{-1},w^{-1},v^{-1}} = \gamma_{w,u,v}$.

(c) Let d be in $\mathcal{D}$. Then $\gamma_{w,d,u} \neq 0$ if and only if $w = u$ and $w \underset{L}{\sim} d$. Moreover $\gamma_{w,d,w} = \gamma_{d,w^{-1},w^{-1}} = \gamma_{w,w^{-1},d} = 1$.

(d) $w \underset{L}{\sim} u^{-1}$ if and only if $\gamma_{w,u,v}$ is not equal to 0 for some v.

(e) If $w \underset{LR}{\leq} u$ then $a(w) \geq a(u)$. In particular if $w \underset{LR}{\sim} u$ then $a(w) = a(u)$. If $w \underset{L}{\leq} u$ (resp. $w \underset{R}{\leq} u$; $w \underset{LR}{\leq} u$) and $a(w) = a(u)$, then $w \underset{L}{\sim} u$ (resp. $w \underset{R}{\sim} u$; $w \underset{LR}{\sim} u$).

(f) The positivity: $\gamma_{w,u,v} \geq 0$ for all w, u, v in W.

1.4. Star operation

Let (W, S) be as in section 1.3. There is a very useful operation on W, called star operation, introduced by Kazhdan and Lusztig in [KL1]. In this section we study the relations between star operation and the integers $\gamma_{w,u,v}$. The main result is Theorem 1.4.5.

1.4.1. Let s and t be in S such that st has order 3, i.e. $sts = tst$. Define

$$D_L(s,t) = \{w \in W \mid L(w) \cap \{s,t\} \text{ has exactly one element}\},$$

$$D_R(s,t) = \{w \in W \mid R(w) \cap \{s,t\} \text{ has exactly one element}\}.$$

If w is in $D_L(s,t)$, then $\{sw, tw\}$ contains exactly one element in $D_L(s,t)$, denoted by ${}^*w$, here $* = \{s,t\}$. The map: $D_L(s,t) \to D_L(s,t)$, $w \to {}^*w$, is an involution and is called a **left star operation**. Similarly if $w \in D_R(s,t)$ we can define the **right star operation** $w \to w^* = \{ws, wt\} \cap D_R(s,t)$ on $D_R(s,t)$, where $* = \{s,t\}$. The following are some properties proved in [KL1].

Let $* = \{s,t\}$. Denote by $< s,t >$ the subgroup of W generated by s and t. Assume that y, w are in $D_L(s,t)$. We have

(a) If yw^{-1} is not in $< s,t >$, then $y \prec w$ if and only if ${}^*y \prec {}^*w$. Moreover $\mu(y,w) = \mu({}^*y, {}^*w)$.

(b) If yw^{-1} is in $< s, t >$, then $y \prec w$ if and only if ${}^*w \prec {}^*y$. Moreover $\mu(y, w) = \mu({}^*w, {}^*y) = 1$.

(c) $y \underset{R}{\sim} w$ if and only if ${}^*y \underset{R}{\sim} {}^*w$.

(d) $w \underset{L}{\sim} {}^*w$.

Let $* = \{s, t\}$. Assume that y, w are in $D_R(s, t)$. We have

(e) If $y^{-1}w$ is not in $< s, t >$, then $y \prec w$ if and only if $y^* \prec w^*$. Moreover $\mu(y, w) = \mu(y^*, w^*)$.

(f) If $y^{-1}w$ is in $< s, t >$, then $y \prec w$ if and only if $w^* \prec y^*$. Moreover $\mu(y, w) = \mu(w^*, y^*) = 1$.

(g) $y \underset{L}{\sim} w$ if and only $w^* \underset{L}{\sim} y^*$.

(h) $w \underset{R}{\sim} w^*$.

Finally we have

(i) If $\Gamma \subseteq D_R(s, t)$ is a left cell of W, then $\Gamma^* = \{w^* \mid w \in \Gamma\}$ is a left cell of W, here $* = \{s, t\}$.

The following lemma shows that C_{w^*} and $C_{{}^*w}$ have nice relationship with C_w.

Lemma 1.4.2. *Let s, t be in S such that st has order 3. Set $* = \{s, t\}$.*
(a) *Assume that w is in $D_L(s, t)$ and $s\,{}^*w \leq {}^*w$. Then*

$$C_s C_w = C_{{}^*w} + \sum_{\substack{y - w \\ sy < y \\ ty < y}} \mu(y, w) C_y.$$

(b) *Assume that w is in $D_R(s, t)$ and $w^*\,s \leq w^*$. Then*

$$C_w C_s = C_{w^*} + \sum_{\substack{y - w \\ ys < y \\ yt < y}} \mu(y, w) C_y.$$

Proof. (a) Since $s\,{}^*w \leq {}^*w$, we have $sw \geq w$ and $tw \leq w$. Thus (see [KL1])

$$C_s C_w = C_{sw} + \sum_{\substack{y \prec w \\ sy < y}} \mu(y, w) C_y.$$

If $w = tw'$ with $sw' \geq w'$ and $tw' \geq w'$, then ${}^*w = sw$. By 1.1 (a), for $y \prec w$ with $sy < y$ we have $ty < y$. In this case (a) is true. If $w = tsw'$ with $sw' \geq w'$ and $tw' \geq w'$, then ${}^*w = sw'$ and $sw = stsw'$. Moreover, by 1.1 (a), if $y \neq {}^*w$, $y \prec w$ and $sy < y$ then we have $ty < y$. In this case (a) is also true. (a) is proved. The proof of (b) is similar.

Lemma 1.4.3. *Let s, t, s', t' be in S such that both st and $s't'$ have order 3. Assume that w is in $D_L(s, t) \cap D_R(s', t')$. Set $* = \{s, t\}$ and $\star = \{s', t'\}$. Then*
(a) *${}^*w$ is in $D_R(s', t')$ and $w^\star$ is in $D_L(s, t)$.*
(b) *${}^*(w^\star) = ({}^*w)^\star$. We shall write ${}^*w^\star$ for ${}^*(w^\star) = ({}^*w)^\star$.*

Proof. (a) By 1.4.1 (d), $w \underset{L}{\sim} {}^*w$, thus we have $R(w) = R({}^*w)$ (see 1.3 (a)), so ${}^*w$ is in $D_R(s', t')$. Similarly we see that $w^\star$ is in $D_L(s, t)$.

(b) By (a), $({}^*w)^\star$ and ${}^*(w^\star)$ are well defined. Since $({}^*w)^\star \underset{R}{\sim} {}^*w$ and ${}^*(w^\star) \underset{L}{\sim} w^\star$, we have

(1) $L({}^*w) = L(({}^*w)^\star)$ and $R(w^\star) = R({}^*(w^\star))$.

Thus we have

(2) Both $({}^*w)^\star$ and ${}^*(w^\star)$ are in $D_L(s, t) \cap D_R(s', t')$.

It is no harm to assume that $s\,{}^*w \leq {}^*w$ and $w^\star\, s' \leq w^\star$. Then $sw \geq w$ and $ws' \geq w$. We also have $sw^\star \geq w^\star$ and ${}^*ws' \geq {}^*w$ since $w^\star \underset{R}{\sim} w$ and ${}^*w \underset{L}{\sim} w$. Write

$$(C_sC_w)C_{s'} = \sum_{y \in W} h_yC_y, \qquad h_y \in \mathcal{A}.$$

By Lemma 1.4.2 and 1.2 (b-c), we get

(3) $h_{({}^*w)^\star} \neq 0$. Moreover, if $y \neq ({}^*w)^\star$ and $h_y \neq 0$, then y is not in $D_L(s, t)$ or y is not in $D_R(s', t')$.

Noting that $C_s(C_wC_{s'}) = (C_sC_w)C_{s'}$, using Lemma 1.4.2 again we see $h_{{}^*(w^\star)}$ is not equal to 0. Now using (2) and (3) we get ${}^*(w^\star) = ({}^*w)^\star$. (b) is proved.

The lemma is proved.

Proposition 1.4.4. *Let s, t, s', t' be as in Lemma 1.4.3. Set $* = \{s, t\}$ and $\star = \{s', t'\}$. Suppose that w is in $D_L(s, t)$ and u is in $D_R(s', t')$. Let v be in W such that $v \underset{L}{\sim} u$ and $v \underset{R}{\sim} w$. Then*

(a) *We have $v \in D_L(s, t) \cap D_R(s', t')$, so we can define ${}^*v^\star$.*

(b) *$h_{w,u,v} = h_{{}^*w,u^\star,{}^*v^\star}$, see 1.2 for the definition of $h_{w,u,v}$.*

Proof. (a) is trivial.

(b) We have ${}^*v^\star \underset{L}{\sim} v^\star$ and ${}^*v^\star \underset{R}{\sim} {}^*v$. By definition we have

(1) $C_wC_u = \displaystyle\sum_{z \in W} h_{w,u,z}C_z, \quad h_{w,u,z} \in \mathcal{A}.$

We may assume that $sw \geq w$ and $us' \geq u$. Using Lemma 1.4.2 we get

(2) $C_sC_w = C_{{}^*w} + \displaystyle\sum_{\substack{y \neq {}^*w \\ sy<y \\ ty<y}} h_yC_y, \quad h_y \in \mathcal{A}.$

(3) $C_uC_{s'} = C_{u^\star} + \displaystyle\sum_{\substack{x \neq u^\star \\ xs'<y \\ xt'<y}} h'_xC_x, \quad h'_x \in \mathcal{A}.$

Write

(4) $C_sC_zC_{s'} = \displaystyle\sum_{z' \in W} f_{z,z'}C_{z'}, \quad f_{z,z'} \in \mathcal{A}.$

Then we have

(5) $$C_sC_wC_uC_{s'} = \sum_{z'\in W}(h_{^*w,u^\star,z'} + \sum_{\substack{y\neq {^*w}\\ sy<y\\ ty<y}} h_yh_{y,u^\star,z'} + \sum_{\substack{x\neq u^\star\\ xs'<y\\ xt'<y}} h'_xh_{^*w,x,z'} + \sum_{\substack{y\neq {^*w}\\ sy<y\\ ty<y}}\sum_{\substack{x\neq u^\star\\ xs'<y\\ xt'<y}} h_yh'_xh_{y,x,z'})C_{z'}.$$

(6) $$C_s\sum_{z\in W}h_{w,u,z}C_zC_{s'} = \sum_{z'\in W}\sum_{z\in W}h_{w,u,z}f_{z,z'}C_{z'}.$$

Using (1) and comparing (5) with (6), we get

(7) $$h_{^*w,u^\star,z'} + \sum_{\substack{y\neq {^*w}\\ sy<y\\ ty<y}} h_yh_{y,u^\star,z'} + \sum_{\substack{x\neq u^\star\\ xs'<y\\ xt'<y}} h'_xh_{^*w,x,z'} + \sum_{\substack{y\neq {^*w}\\ sy<y\\ ty<y}}\sum_{\substack{x\neq u^\star\\ xs'<y\\ xt'<y}} h_yh'_xh_{y,x,z'} = \sum_{z\in W}h_{w,u,z}f_{z,z'}.$$

Using 1.2 (a-b) we see

(8) Assume that $z' \underset{R}{\sim} {^*w}$ and $z' \underset{L}{\sim} u^\star$. Then

$$\sum_{\substack{y\neq {^*w}\\ sy<y\\ ty<y}} h_yh_{y,u^\star,z'} + \sum_{\substack{x\neq u^\star\\ xs'<y\\ xt'<y}} h'_xh_{^*w,x,z'} + \sum_{\substack{y\neq {^*w}\\ sy<y\\ ty<y}}\sum_{\substack{x\neq u^\star\\ xs'<y\\ xt'<y}} h_yh'_xh_{y,x,z'} = 0.$$

When $h_{w,u,v} = 0$ we must have $h_{^*w,u^\star,^*v^\star} = 0$. Otherwise, noting that ${^*v^\star} \underset{R}{\sim} {^*w}$ and ${^*v^\star} \underset{L}{\sim} u^\star$, by (7) and (8) we have $h_{^*w,u^\star,^*v^\star} = \sum_{z\in W} h_{w,u,z}f_{z,^*v^\star}$ and $h_{w,u,z}f_{z,^*v^\star} \neq 0$ for some z. Assume that $h_{w,u,z}f_{z,^*v^\star} \neq 0$. By 1.2 (c) and 1.3 (e) we have $z \underset{R}{\sim} w$ and $z \underset{L}{\sim} u$. By the proof of Lemma 1.4.3 we then have $f_{z,^*z^\star} = 1$, and $z'' \underset{R}{\not\sim} {^*z}$, $z'' \underset{L}{\not\sim} z^\star$ if $f_{z,z''} \neq 0$ and $z'' \neq {^*z^\star}$. Thus ${^*v^\star} = {^*z^\star}$ and $v = z$. This contradicts that $h_{w,u,v} = 0$. Therefore $h_{^*w,u^\star,^*v^\star} = 0$ whenever $h_{w,u,v} = 0$.

Now suppose that $h_{w,u,v} \neq 0$. As above we have $z = v$ whenever $h_{w,u,z}f_{z,^*v^\star} \neq 0$, and $f_{v,^*v^\star} = 1$. Using (7) and (8) we then get $h_{^*w,u^\star,^*v^\star} = h_{w,u,v}$ in this case. We proved (b).

The Proposition is proved.

Theorem 1.4.5. *Suppose that w is in $D_L(s,t)\cap D_R(s'',t'')$ and u is in $D_L(s'',t'')\cap D_R(s',t')$. Set $* = \{s,t\}, \# = \{s'',t''\}$, and $\star = \{s',t'\}$. Let v be in W such that v is in $D_L(s,t)\cap D_R(s',t')$. Then we have*

$$\gamma_{w,u,v} = \gamma_{^*w^\#,^\#u^\star,^*v^\star}.$$

Proof. By Lemma 1.4.3 and Proposition 1.4.4 (a), the elements ${^*w^\#}, {^\#u^\star}$ and ${^*v^\star}$ are well defined. If either $v \underset{R}{\not\sim} w$ or $v \underset{L}{\not\sim} u$, then both sides of the wanted equality are 0. Now suppose that $v \underset{R}{\sim} w$ and $v \underset{L}{\sim} u$. According to Proposition 1.4.4

(b) we have $h_{w,u,v} = h_{{}^*w,u^*,{}^*v^*}$. Therefore $\gamma_{w,u,v} = \gamma_{{}^*w,u^*,{}^*v^*}$. It is easy to see that $({}^*w)^{-1} = (w^{-1})^*$, ${}^\#(w^{-1})^* = ({}^*w^\#)^{-1}$, and ${}^\#(({}^*w)^{-1}) = ({}^*w^\#)^{-1}$. Thus we have

$$\begin{aligned}
\gamma_{w,u,v} &= \gamma_{{}^*w,u^*,{}^*v^*} && \text{using 1.3 (b)}\\
&= \gamma_{u^*,({}^*v^*)^{-1},({}^*w)^{-1}} && \text{using Prop. 1.4.4}\\
&= \gamma_{{}^\#u^*,({}^*v^*)^{-1},{}^\#(({}^*w)^{-1})}\\
&= \gamma_{{}^\#u^*,({}^*v^*)^{-1},({}^*w^\#)^{-1}} && \text{using 1.3 (b)}\\
&= \gamma_{{}^*w^\#,{}^\#u^*,{}^*v^*}.
\end{aligned}$$

The theorem is proved.

Proposition 1.4.6. *Let W be an extended affine Weyl group.*

(a) *Let I be a subset of S such that the subgroup W_I of W generated by I is finite. Then the longest element w_I is a distinguished involution.*

In (b) and (c) d is a distinguished involution.

(b) *For any $\omega \in \Omega$, the element $\omega d\omega^{-1}$ is a distinguished involution.*

(c) *Suppose $s, t \in S$ and st has order 3. Then $d \in D_L(s,t)$ if and only if $d \in D_R(s,t)$. If $d \in D_L(s,t)$, then ${}^*d^*$ is a distinguished involution.*

Proof. (a) is well known, see for example [L5].

(b) Since $C_\omega C_w C_{\omega^{-1}} = C_{\omega w\omega^{-1}}$ and $C_\omega C_{\omega^{-1}} = 1$, for any w, u, v in W we have

$$P_{\omega u\omega^{-1},\omega w\omega^{-1}} = P_{u,w}$$

and

$$h_{\omega w\omega^{-1},\omega u\omega^{-1},\omega v\omega^{-1}} = h_{w,u,v}.$$

In particular we have $a(\omega w\omega^{-1}) = a(w)$ and $P_{e,\omega w\omega^{-1}} = P_{e,w}$ for any $w \in W$. Noting that $l(\omega w\omega^{-1}) = l(w)$ for any w in W, we see

$$\deg P_{e,\omega d\omega^{-1}} = \deg P_{e,d} = \frac{1}{2}(l(d) - a(d)) = \frac{1}{2}(l(\omega d\omega^{-1}) - a(\omega d\omega^{-1})).$$

By definition, $\omega d\omega^{-1}$ is a distinguished involution.

(c) Let d' be the distinguished involution of the left cell containing ${}^*d^*$. We have

$$({}^*d^*)^{-1} = {}^*(d^{-1})^* = {}^*d^*,$$

so ${}^*d^*$ is an involution. Using 1.3 (c-d) we get $\gamma_{{}^*d^*,d',{}^*d^*} = 1$. Using Theorem 1.4.5 we then have $\gamma_{d,{}^*d'^*,d} = 1$. Using 1.3 (b) we get $\gamma_{{}^*d'^*,d,d} = 1$. Applying 1.3 (c) we have $d = {}^*d'^*$. Therefore $d' = {}^*d^*$.

The proposition is proved.

1.5. Based ring

Following [L5] we define the based rings. Let J be the free $\mathbb{Z}$-module with a basis $\{t_w \mid w \in W\}$. The multiplication $t_w t_u = \sum_{v\in W} \gamma_{w,u,v} t_v$ defines an associative ring structure on J, see [L5]. The ring J is called the **based ring** of W, its unit is $\sum_{d\in\mathcal{D}} t_d$. According to 1.3(a), for each two-sided cell $\mathbf{c}$ of W, the $\mathbb{Z}$-submodule $J_{\mathbf{c}}$ of J, spanned by all t_w ($w \in \mathbf{c}$), is a two-sided ideal of J. The ideal $J_{\mathbf{c}}$ is in fact an associative ring with unit $\sum_{d\in\mathcal{D}\cap\mathbf{c}} t_d$. The ring $J_{\mathbf{c}}$ is called the **based ring of the two-sided cell c**. For a left cell Γ of W, the $\mathbb{Z}$-submodule $J_{\Gamma\cap\Gamma^{-1}}$ of J, spanned by all t_w ($w \in \Gamma\cap\Gamma^{-1}$), is also an associative ring, its unit is t_d, here d is the unique distinguished involution in Γ.

Proposition 1.5.1. *Let Γ be a left cell of W.*
(a) *For any $\omega \in \Omega$, $\Gamma' = \omega\Gamma\omega^{-1}$ is a left cell of W. Moreover the map $w \to \omega w\omega^{-1}$ induces an isomorphism between the based rings $J_{\Gamma\cap\Gamma^{-1}}$ and $J_{\Gamma'\cap\Gamma'^{-1}}$.*
(b) *Let $s, t \in S$ be such that st has order 3. Suppose $\Gamma \subset D_R(s,t)$. Then $J_{\Gamma\cap\Gamma^{-1}} \simeq J_{\Gamma^*\cap(\Gamma^*)^{-1}}$, here $* = \{s,t\}$.*

Proof. (a) Obviously Γ' is a left cell of W. From the proof of Prop. 1.4.6 (b) we see $\gamma_{\omega w\omega^{-1},\omega u\omega^{-1},\omega v\omega^{-1}} = \gamma_{w,u,v}$ for any $w,u,v \in W$. Therefore the map $w \to \omega w\omega^{-1}$ induces an isomorphism between the based rings $J_{\Gamma\cap\Gamma^{-1}}$ and $J_{\Gamma'\cap\Gamma'^{-1}}$.

(b) Suppose $w \in \Gamma\cap\Gamma^{-1}$. Then ${}^*w^*$ is in $\Gamma^*\cap(\Gamma^*)^{-1}$. The map $w \to {}^*w^*$ defines a bijection between $\Gamma\cap\Gamma^{-1}$ and $\Gamma^*\cap(\Gamma^*)^{-1}$. According to Theorem 1.4.5 we see that the map $t_w \to t_{{}^*w^*}$ defines a ring isomorphism between $J_{\Gamma\cap\Gamma^{-1}}$ and $J_{\Gamma^*\cap(\Gamma^*)^{-1}}$.

The proposition is proved.

The based rings $J_{\mathbf{c}}$ are very interesting in understanding the structure and representations of Hecke algebras of W, see [L5-L8]. If we know the structure of $J_{\mathbf{c}}$ explicitly we can construct modules of affine Hecke algebras in a computable way, see [L8, X3].

Lusztig has a nice conjecture concerning the structure of $J_{\mathbf{c}}$. Assume that G is connected and has a simply connected derived group. There is a natural bijection between the set of two-sided cells of W and the set of unipotent classes of G, see [L8]. Assume that u is an element in the unipotent class corresponding to a two-sided cell $\mathbf{c}$ of W. Denote by $F_{\mathbf{c}}$ a maximal reductive subgroup of the centralizer $C_G(u)$ of u in G. Lusztig conjectured that there exists a finite $F_{\mathbf{c}}$-set Y and a bijection $\pi : \mathbf{c} \to$ the set of isomorphism classes of irreducible $F_{\mathbf{c}}$ vector bundles on $Y\times Y$ such that $t_w \to \pi(w)$ defines a ring isomorphism (preserving the unit element) between $J_{\mathbf{c}}$ and $K_{F_{\mathbf{c}}}(Y\times Y)$ and $\pi(w^{-1}) = \widetilde{\pi(w)}$, see [L8] for the conjecture and for the definition of $\widetilde{\pi(w)}$.

When $F_{\mathbf{c}}$ is connected (or equivalently $C_G(u)$ is connected), $F_{\mathbf{c}}$ must act on Y trivially. In this case $|Y|$ is the number of left cells contained in $\mathbf{c}$ and $K_{F_{\mathbf{c}}}(Y\times Y)$ is isomorphic to the $|Y|\times|Y|$ matrix algebra $M_{|Y|}(R_{F_{\mathbf{c}}})$ over the rational representation ring $R_{F_{\mathbf{c}}}$ of $F_{\mathbf{c}}$. Let $\mathrm{Irr}F_{\mathbf{c}}$ be the set of isomorphism classes of rational irreducible representations of $F_{\mathbf{c}}$. Then Lusztig Conjecture says that there is a bijection

$$\pi : \mathbf{c} \to \{(V,i,j) \mid V \in \mathrm{Irr}F_{\mathbf{c}},\ 1 \le i,j \le |Y|\}$$

such that (1) the map $t_w \to \pi(w)$ defines a ring isomorphism between $J_{\mathbf{c}}$ and $M_{|Y|}(R_{F_{\mathbf{c}}})$, where we identify $\pi(w) = (V, i, j)$ with the matrix whose (i, j)-entry is V and other entries are 0, (2) $\pi(w^{-1}) = (V^*, j, i)$ if $\pi(w) = (V, i, j)$, here V^* is the dual of V.

Lusztig Conjecture has been proved for the following cases, (1) $\mathbf{c}$ is the lowest two-sided cell of W, (that is, $\mathbf{c}$ is the two-sided cell of W containing the longest element w_0 of W_0,) see [X1], (2) rank 2 cases, see [X3], (3) the case $a(\mathbf{c}) = 1$, see [X3].

When $G = GL_n(\mathbb{C})$, each $F_{\mathbf{c}}$ is connected. The purpose of this paper is to show that Lusztig Conjecture is true for the extended affine Weyl group associated with $GL_n(\mathbb{C})$.

1.6. Star operation, II

The star operation, introduced in [KL1], was generalized in [L4]. We are interested in the relationship between the (generalized) star operation and the structure constants $\gamma_{w,u,v}$ of the based ring of an extended affine Weyl group. In this section we show that Theorem 1.4.5 remains true for the generalized star operation. We also give a few other identities about the constants $\gamma_{w,u,v}$. The results in this section are not used in the sequent chapters but should be useful for understanding the based ring of an arbitrary extended affine Weyl group.

Let (W, S) be as in section 1.3. Assume that s and t are in S and st has order m. Denote by U the subgroup of W generated by s and t. Each coset Uw can be decomposed into four parts: one consists of the unique element x of minimal length in the coset, one consists of the unique element y of maximal length in the coset, one consists of the $m-1$ elements $sx,\ tsx,\ stsx,\ ...$, one consists of the $m-1$ elements $tx,\ stx,\ tstx,\ ....$ The last two subsets are called left strings (related to $\{s, t\}$) and shall be regarded as sequences (as above) rather than subsets. Similarly we define right strings (related to $\{s, t\}$). We have (see [L4])

(a) A left string in W is contained in a left cell of W and a right string in W is contained in a right cell of W.

Assume that w is in a left (resp. right) string (related to $\{s, t\}$) of length $m-1$ and is the ith element of the left (resp. right) string, we define ${}^*w$ (resp. w^*) to be the $(m-i)$th element of the string, where $* = \{s, t\}$. We have

Lemma 1.6.1. *Let w be in W such that w is in a left string related to $* = \{s, t\}$ and is also in a right string related to $\star = \{s', t'\}$. Then*

(a) *${}^*w$ is in a right string related to $\{s', t'\}$ and $w^\star$ in a left string related to $\{s, t\}$.*
(b) *${}^*(w^\star) = ({}^*w)^\star$. We shall write ${}^*w^\star$ for ${}^*(w^\star) = ({}^*w)^\star$.*

The proof is similar to that of Lemma 1.4.3 although more complicated.

The following theorem is a generalization of Theorem 1.4.5

Theorem 1.6.2. *Let w, u, v be in W such that (1) w is in a left string related to $* = \{s, t\}$ and also in a right string related to $\# = \{s', t'\}$, (2) u is in a left string related to $\# = \{s', t'\}$ and also in a right string related to $\star = \{s'', t''\}$, (3) v is in a left string related to $* = \{s, t\}$ and also in a right string related to $\star = \{s'', t''\}$. Then*

$$\gamma_{w,u,v} = \gamma_{{}^*w^\#,{}^\#u^\star,{}^*v^\star}.$$

The proof is similar to that of Theorem 1.4.5.

1.6.3. In this subsection we assume that s and t are in S and st has order 4. Let w, u, v be in W such that $l(ststw) = 4 + l(w)$ and $l(ststv) = 4 + l(v)$. As in the proof of Lemma 1.4.2 we have

$$C_tC_{sw} = C_{tsw} + \sum_{\substack{y-w\\ ty<y\\ sy<y}} \mu(y,w)C_y,$$

$$C_tC_{stw} = C_{tstw} + C_{tw} + \sum_{\substack{y-w\\ ty<y\\ sy<y}} \mu(y,w)C_y,$$

$$C_tC_{stsw} = C_{tsw} + \sum_{\substack{y-w\\ ty<y\\ sy<y}} \mu(y,w)C_y.$$

Considering the products $C_tC_{sw}C_u$, $C_tC_{stw}C_u$, $C_tC_{stsw}C_u$, and using the associativity of multiplication of the Hecke algebra H, we get

(a) $h_{tsw,u,tv} = h_{sw,u,stv}$,

(b) $h_{tsw,u,tsv} = h_{sw,u,sv} + h_{sw,u,stsv}$,

(c) $h_{tsw,u,tstv} = h_{sw,u,stv}$,

(d) $h_{tstw,u,tv} + h_{tw,u,tv} = h_{stw,u,stv}$,

(e) $h_{tstw,u,tsv} = h_{stw,u,stsv}$,

(f) $h_{tstw,u,tstv} + h_{tw,u,tstv} = h_{stw,u,stv}$.

In particular, we have

(a') $\gamma_{tsw,u,tv} = \gamma_{sw,u,stv}$,

(b') $\gamma_{tsw,u,tsv} = \gamma_{sw,u,sv} + \gamma_{sw,u,stsv}$,

(c') $\gamma_{tsw,u,tstv} = \gamma_{sw,u,stv}$,

(d') $\gamma_{tstw,u,tv} + \gamma_{tw,u,tv} = \gamma_{stw,u,stv}$,

(e') $\gamma_{tstw,u,tsv} = \gamma_{stw,u,stsv}$,

(f') $\gamma_{tstw,u,tstv} + \gamma_{tw,u,tstv} = \gamma_{stw,u,stv}$.

One may compare the above equalities with the equalities in [L4, (10.4.2)]. Using 1.3 (b) and the above equalities we can get more equalities for the structure constants of the based ring J, we omit them. When st has higher order, there exist similar equalities. We omit the discussion.

CHAPTER 2

Type $\tilde{A}_{n-1}$

From now on we will concentrate on type $\tilde{A}_{n-1}$. The cells of an affine Weyl group of type $\tilde{A}_{n-1}$ have been described in [S, L3]. In this chapter we first recall some facts about the cells, then we derive some new results for our purpose. In section 2.1 we recall an alternative definition (due to Lusztig) for the extended affine Weyl group W associated with $GL_n(\mathbb{C})$. In section 2.2 we recall the description of cells of W in [S, L3]. For this description and later use we slightly refine the definition of chain and antichain in [S]. More precisely we will define d-chain (resp. d-antichain) and r-chain (resp. r-antichain). In this section we also consider the intersection of left cells and right cells. The intersections are important to our purpose. In section 2.3 we show that the based ring of a two-sided cell of W is isomorphic to a matrix algebra over the based ring $J_{\Gamma\cap\Gamma^{-1}}$ for any left cell Γ in the two-sided cell (see Theorem 2.3.2). This is the main result of this chapter. Thus to understand Lusztig's conjecture on the structure of the based ring of the two-sided cell we only need to understand the structure of $J_{\Gamma\cap\Gamma^{-1}}$. We will do this in Chapters 5-8.

For later use in section 2.4 we work out some results about chains and antichains. The results will be needed in Chapter 5 for defining the required bijection between $\Gamma\cap\Gamma^{-1}$ and $\mathrm{Irr}F_{\mathbf{c}}$, see Chapter 5. In section 2.5 we prove a result about star operation.

2.1. The affine Weyl group associated with $GL_n(\mathbb{C})$

In this section we recall a definition of Lusztig for affine Weyl group of type $\tilde{A}_{n-1}$, see [L2].

Suppose that $G = GL_n(\mathbb{C})$ is the general linear group over $\mathbb{C}$ of degree n. Then the Weyl group W_0 of G is isomorphic to the symmetric group of n letters. Let T be the subgroup of G consisting of all diagonal matrices in G. Then T is a maximal torus of G and the weight lattice X is the character group of T. Let $\tau_i \in X$ be the homomorphism $T \to \mathbb{C}^*$, $\mathrm{diag}(a_1, ..., a_n) \to a_i$. We have $X =< \tau_1, ..., \tau_n >$ and the extended affine Weyl group associated with $GL_n(\mathbb{C})$ is $W = W_0 \ltimes X$. We identify W_0 with the group of all the $n\times n$ permutation matrices (by a permutation matrix, we mean a monomial matrix whose nonzero entries are all equal to 1). Let s_i ($i = 1, 2, ..., n-1$) be the simple reflection of W_0 obtained from the identity matrix by interchanging the ith and the $(i+1)$th rows. Then $s_i\tau_i = \tau_{i+1}s_i$ and $s_i\tau_j = \tau_j s_i$ if $j \neq i, i+1$.

Another realization of W, due to Lusztig, is as follows. Consider the permutations $\sigma : \mathbb{Z} \to \mathbb{Z}$ that satisfy $\sigma(i+n) = \sigma(i)+n$ and $\sum_{i=1}^{n}(\sigma(i)-i) \equiv 0(\mathrm{mod}\ n)$. All such permutations form a permutation group W_* of $\mathbb{Z}$.

Lemma 2.1.1: *W is isomorphic to W_*.*

Proof. We can define a W-action on $\mathbb{Z}$ as follows,

$$s_i(j) = \begin{cases} j & \text{if } j \not\equiv i, i+1 (\text{mod } n), \\ j+1 & \text{if } j \equiv i (\text{mod } n), \\ j-1 & \text{if } j \equiv i+1 (\text{mod } n); \end{cases}$$

$$\tau_i(j) = \begin{cases} j & \text{if } j \not\equiv i (\text{mod } n), \\ j+n & \text{if } j \equiv i (\text{mod } n). \end{cases}$$

It is easy to check that this action defines an isomorphism between W and W_*. The lemma is proved.

We shall identify W and W_* using the above isomorphism.

Lemma 2.1.2. *The permutation $\omega : \mathbb{Z} \to \mathbb{Z}, i \to i+1$ for all i, is in W.*

Proof. It is clear from the definition.

2.1.3. The following are some simple properties of the extended affine Weyl group W.

(a) The center of W is generated by ω^n.

(b) $\omega s_i = s_{i+1}\omega$, where s_0 is defined by $s_0(0) = 1, s_0(1) = 0, s_0(i) = i$ if $i \not\equiv 0, 1 (\text{mod } n)$, and $s_i = s_j$ if $i \equiv j (\text{mod } n)$.

(c) Let Ω be the subgroup of W generated by ω and W' the subgroup of W generated by all s_i. Then Ω is an infinite cyclic group and W' is an affine Weyl group of type $\tilde{A}_{n-1}$. Moreover we have $W \simeq \Omega \ltimes W'$.

(d) We have $\tau_i = \omega s_{i-2} s_{i-3} \cdots s_0 s_{n-1} \cdots s_i$ if $i \geq 2$ and $\tau_1 = \omega s_{n-1} s_{n-2} \cdots s_1$. Note that the length of τ_i is $n-1$.

(e) As a special case of the length formula in [IM], we have

$$l(w) = \sum_{1 \leq i < j \leq n} \left| \left[\frac{w(j) - w(i)}{n} \right] \right|,$$

where [h] is the integer part of h (recall that $0 \leq h - [h] < 1$). For a direct proof of the formula, see [S, Lemma 4.2.2].

(f) Let $w \in W$. Then $w(k) < w(k+1)$ if and only if $w \leq w s_k$, see [S, Corollary 4.2.3].

2.2. Cells

The cells in W have a beautiful combinatorial description, conjectured by Lusztig and proved by Shi. In this section we recall the description of cells of W in [S, L3].

Following Shi we define chains and antichains. More precisely we refine slightly his definition by defining d-chains, d-antichains, r-chains and r-antichains.

Let w be an element of W and $j_1, j_2, \cdots, j_k$ be integers. We call $j_1, j_2, \cdots, j_k$ a **d-chain** of w of length k and $w(j_1), ..., w(j_k)$ an **r-chain** of w of length k if
(1) $j_1 < j_2 < \cdots < j_k$,
(2) $j_i \not\equiv j_{i'} (\mathrm{mod}\ n)$ whenever $1 \leq i \neq i' \leq k$,
(3) $w(j_1) > w(j_2) > \cdots > w(j_k)$,
here d-chain means domain chain and r-chain means range chain. Thus a d-chain of W essentially is a subset of $\mathbb{Z}$ whose natural order is reversed by w.

A **d-chain (resp. r-chain) family set** of w of index q is a subset Y of $\mathbb{Z}$ such that (1) elements in Y are noncongruent to each other modulo n, (2) Y is a disjoint union of q d-chains (resp. r-chains) $A_1, ..., A_q$ of w. We also call $\{A_1, ..., A_q\}$ a **d-chain (resp. r-chain) family** of W.

We call $j_1, j_2, \cdots, j_k$ a **d-antichain** of w of length k and $w(j_1), ..., w(j_k)$ an **r-antichain** of w of length k if
(1) $j_k - n < j_1 < j_2 < \cdots < j_k$, (then $j_i \not\equiv j_{i'} (\mathrm{mod}\ n)$ whenever $1 \leq i \neq i' \leq k$,)
(2) $w(j_k) - n < w(j_1) < w(j_2) < \cdots < w(j_k)$.

A **d-antichain (resp. r-antichain) family set** of w of index q is a subset Y of $\mathbb{Z}$ such that (1) elements in Y are noncongruent to each other modulo n, (2) Y is a disjoint union of q d-antichains (resp. r-antichains) $A_1, ..., A_q$ of w. We also call $\{A_1, ..., A_q\}$ a **d-antichain (resp. r-antichain) family** of W.

Obviously if Y is an r-chain (resp. r-antichain) family set of w of index q, then Y is a d-chain (resp. d-antichain) family set of w^{-1} of index q.

We shall regard W as the permutation group W_* of $\mathbb{Z}$. Following Lusztig we associate a partition λ of n with an element w of W as follows. Let d_i be the maximal one among the cardinalities of all d-chain family sets of w of index i. Then $d_1 \leq d_2 \leq \cdots \leq d_n = n$. According to [G, Th. 1.5], $d_1 \geq d_2 - d_1 \geq d_3 - d_2 \geq \cdots \geq d_n - d_{n-1}$. We call

$$\lambda(w) = (d_1, d_2 - d_1, ..., d_n - d_{n-1})$$

the **partition associated with** w. According to [S, L3], $w \underset{LR}{\sim} u$ if and only if $\lambda(w) = \lambda(u)$. Moreover the number of left cells in a two-sided cell corresponding to a partition λ of n is $n_\mu = n!/(\mu_1!\mu_2! \cdots \mu_{r'}!)$, where $(\mu_1, ..., \mu_{r'})$ is the dual partition of λ, see [S].

The dual partition of λ can be defined through antichains of w. Let d'_i be the maximal one among the cardinalities of all d-antichain family sets of w of index i. Then $d'_1 \leq d'_2 \leq \cdots \leq d'_n = n$. According to [G, Th. 1.5], $d'_1 \geq d'_2 - d'_1 \geq d'_3 - d'_2 \geq \cdots \geq d'_n - d'_{n-1}$. The partition

$$\mu(w) = (d'_1, d'_2 - d'_1, ..., d'_n - d'_{n-1})$$

is the partition dual to the partition $\lambda(w)$, see [G, Th. 1.6].

Let $\mathbf{c}$ be the two-sided cell of W corresponding to a partition $\lambda = (\lambda_1, ..., \lambda_r)$ of n. Denote by w_λ the longest element of the subgroup of W generated by all

$$s_1,\ ...,\ s_{\lambda_1 - 1},\ s_{\lambda_1+1},\ ...,\ s_{\lambda_1+\lambda_2-1},\ ...,\ s_{\lambda_1+\cdots+\lambda_{r-1}+1},\ ...,\ s_{\lambda_1+\cdots+\lambda_r-1}.$$

Then $w_\lambda \in \mathbf{c}$ and there is a unique left cell in $\mathbf{c}$ containing w_λ, denoted by Γ_λ. We shall write $\Gamma_{\lambda,i}$ for the left cell containing $\omega^i w_\lambda \omega^{-i}$. Set $\Phi_{\lambda,i} = \Gamma_{\lambda,i}^{-1}$, this is a right cell contained in $\mathbf{c}$.

Denote by u_λ the longest element of the subgroup of W generated by all

$$s_1,\ ...,\ s_{\lambda_r - 1},\ s_{\lambda_r+1},\ ...,\ s_{\lambda_r+\lambda_{r-1}-1},\ ...,\ s_{\lambda_r+\cdots+\lambda_2+1},\ ...,\ s_{\lambda_r+\cdots+\lambda_1-1}.$$

Then $u_\lambda \in \mathbf{c}$ and there is a unique left cell in $\mathbf{c}$ containing u_λ, denoted by Δ_λ. We shall write $\Delta_{\lambda,i}$ for the left cell containing $\omega^i u_\lambda \omega^{-i}$. Set $\Psi_{\lambda,i} = \Delta_{\lambda,i}^{-1}$. Note that the intersection of the set H_λ in [S, 9.3] and $\mathbf{c}$ is contained in the union $\bigcup_{1\le i\le n} \Delta_{\lambda,i}^{-1}$.

Lemma 2.2.1. *Assume that $w \in \mathbf{c}$. Then through a succession of left star operations and of right star operations on w we can get an element in $\Gamma_{\lambda,i} \cap \Phi_{\lambda,j}$ for some integers i, j.*

Proof. It follows from Prop. 9.3.7, Theorem 1.6.3 (i) and Lemma 18.3.2 in [S].

Corollary 2.2.2. *Assume that Γ is a left cell in $\mathbf{c}$ and Φ is a right cell in $\mathbf{c}$. Then $\Gamma \cap \Phi$ can be obtained by applying a succession of left star operations and of right star operations on some $\Gamma_{\lambda,i} \cap \Phi_{\lambda,j}$.*

Since $\Gamma_{\lambda,i} = \Gamma_\lambda \omega^{-i}$, the map $w \to \omega^j w \omega^{-i}$ defines a bijection between $\Gamma_\lambda \cap \Gamma_\lambda^{-1}$ and $\Gamma_{\lambda,i} \cap \Phi_{\lambda,j}$. Thus it is very fundamental to understand the properties of $\Gamma_\lambda \cap \Gamma_\lambda^{-1}$.

Proposition 2.2.3. *Let Γ be a left cell in $\mathbf{c}$. Then there is a bijection $\phi : \Gamma_\lambda \cap \Gamma_\lambda^{-1} \to \Gamma \cap \Gamma^{-1}$ such that $t_w \to t_{\phi(w)}$ defines an isomorphism between $J_{\Gamma_\lambda \cap \Gamma_\lambda^{-1}}$ and $J_{\Gamma\cap\Gamma^{-1}}$.*

Proof. Using Corollary 2.2.2 and Lemma 18.3.2 in [S], we can find i such that $\omega^i(\Gamma \cap \Gamma^{-1})\omega^{-i}$ is obtained from $\Gamma_\lambda \cap \Gamma_\lambda^{-1}$ by applying a succession of left star operations and the corresponding right star operations. Using Prop. 1.5.1 we see that the assertion is true.

2.3. The based ring $J_{\mathbf{c}}$

In this section we will show that the based ring $J_{\mathbf{c}}$ of a two-sided cell $\mathbf{c}$ of W is a matrix algebra over $J_{\Gamma\cap\Gamma^{-1}}$ for any left cell Γ in $\mathbf{c}$. This is the main result of this chapter and is also one of the key steps of our proof of Lusztig Conjecture for the structure of $J_{\mathbf{c}}$.

Recall that for the two-sided cell $\mathbf{c}$ corresponding to a partition λ of n, we have a unique left cell Γ_λ containing w_λ. We number the left cells in $\mathbf{c}$ as $\Gamma_\lambda = \Gamma_1, \Gamma_2, ..., \Gamma_{n_\mu}$, where $n_\mu = n!/(\mu_1!\cdots\mu_{r'}!)$ and $\mu = (\mu_1, \mu_2, ..., \mu_{r'})$ is the dual partition of $\lambda = (\lambda_1, ..., \lambda_r)$. Let $\Phi_i = \Gamma_i^{-1}$ be the right cell corresponding to Γ_i. We would like to define a bijection ϕ_{ij} between $A_{ij} = \Phi_i \cap \Gamma_j$ and $A_{11} = \Phi_1 \cap \Gamma_1 = \Gamma_1^{-1} \cap \Gamma_1$.

There are several cases. First we suppose $j = 1$. Let $l \ge 1$ be such that $\omega^l w_\lambda \omega^{-l} = w_\lambda$ but $\omega^h w_\lambda \omega^{-h} \ne w_\lambda$ for $1 \le h \le l-1$.

(a) If $\Phi_i = \omega^h \Phi_1$ for some $0 \leq h \leq l-1$, then obviously $\omega^h w \to w$ defines a bijection from A_{i1} to A_{11}.
(b) Now let Φ_i be arbitrary. Using [S, Prop.9.3.7], we can find h_i between 0 and $l-1$ such that Φ_i is obtained from $\Phi = \omega^{h_i}\Phi_1$ by a sequence of left star operations. Such h_i and sequence of left star operations are usually not unique. We fix such an h_i and the sequence of left star operations. Then $\Phi_i \cap \Gamma_1$ is obtained from $\Phi \cap \Gamma_1$ by applying the sequence of left star operations. This of course establishes a bijection between $\Phi_i \cap \Gamma_1$ and $\Phi \cap \Gamma_1$. Using (a) we then get a bijection between $\Phi_i \cap \Gamma_1$ and $A_{11} = \Gamma_1^{-1} \cap \Gamma_1$.
(c) Now for any Φ_i we have fixed an ω^{h_i} and a sequence of left star operations. Then Γ_i can be obtained from $\Gamma_1\omega^{-h_i}$ by applying the corresponding sequence of right star operations. By this way and using (a) we get a bijection ϕ_{ij} between A_{ij} and A_{11}, cf. Lemma 2.2.1.

The following are some properties of the bijection $\phi_{ij} : A_{ij} \to A_{11}$.

Lemma 2.3.1. (a) *Let d_i be the distinguished involution in A_{ii}. Then $\phi_{ii}(d_i) = w_\lambda$.*
(b) *Note that $A_{ij}^{-1} = A_{ji}$. For $x \in A_{ji}$ we have $\phi_{ij}(x^{-1}) = (\phi_{ji}(x))^{-1}$.*

Proof. Using Prop. 1.4.6 we see that (a) is true. (b) follows from $({}^*x^\star)^{-1} = {}^\star(w^{-1})^*$ and $(\omega x\omega^{-1})^{-1} = \omega x^{-1}\omega^{-1}$.

We shall use $E(t_w, i, j)$ for any square matrix whose (i,j)-entry is t_w and other entries are 0.

Theorem 2.3.2. *Let $\mathbf{c}$ be the two-sided cell of W corresponding to a partition λ of n and μ the dual partition of λ.*
(a)*The map $t_w \to t_{\phi_{ii}(w)}$ induces a ring isomorphism from $J_{\Gamma_i\cap\Gamma_i^{-1}}$ to $J_{\Gamma_\lambda\cap\Gamma_\lambda^{-1}}$.*
(b) *The based ring $J_{\Gamma_\lambda\cap\Gamma_\lambda^{-1}}$ is commutative.*
(c) *The map*

$$t_w \to E(t_{\phi_{ij}(w)}, i, j), \qquad w \in A_{ij},$$

defines an isomorphism from the based ring $J_{\mathbf{c}}$ to $M_{n_\mu}(J_{\Gamma_\lambda^{-1}\cap\Gamma_\lambda})$, the $n_\mu \times n_\mu$ matrix algebra over the ring $J_{\Gamma_\lambda^{-1}\cap\Gamma_\lambda}$.

Proof. (a) follows from Prop. 1.5.1 and its proof.

(b) Let $\Gamma_{\mathbf{c}}$ be the canonical left cell in $\mathbf{c}$, i.e., $\Gamma_{\mathbf{c}}$ is a left cell in $\mathbf{c}$ and $R(w) \subseteq \{s_0\}$. By [LX] we know that $J_{\Gamma_{\mathbf{c}}\cap\Gamma_{\mathbf{c}}^{-1}}$ is commutative. Using (a) we see that (b) is true.

(c) Note that for any $\omega, \omega', \omega''$ in Ω and x, y, z in $\mathbf{c}$ we have

$$\gamma_{x,y,z} = \gamma_{\omega x\omega'', \omega''^{-1}y\omega', \omega z\omega'}.$$

Let $x \in A_{ij}, y \in A_{jk}$ and $z \in A_{ik}$. Using Theorem 1.4.5 repeatedly we get

$$\gamma_{x,y,z} = \gamma_{\phi_{ij}(x), \phi_{jk}(y), \phi_{ik}(z)}.$$

Combining this and 1.3 (a) we see that (c) is true.

The theorem is proved.

Remark: For Φ_i if we choose different h_i and/or different sequence of left star operations then we would usually get a different isomorphism in Theorem 2.3.2 (c).

For instance, let $n = 3$ and $\mathbf{c}$ the two-sided cell of W corresponding to the partition $\lambda = (2, 1)$. We have $s_\lambda = s_1$. Let Φ_i $(i = 0, 1, 2)$ be the right cell of W containing s_i. Then $\mathbf{c}$ is the union of Φ_0, Φ_1 and Φ_2. We have $\Phi_0 = \omega^2\Phi_1$ and $\Phi_2 = \omega\Phi_1$. Let $* = \{s_1, s_0\}$ and $\star = \{s_1, s_2\}$. We also have $\Phi_0 = {}^*\Phi_1$ and $\Phi_2 = {}^\star\Phi_1$. In this example it is easy to see that different h_i and/or different sequences of left star operations usually lead to different isomorphisms in Theorem 2.3.2 (c).

Let λ be a partition of n and $\mathbf{c}$ the two-sided cell of W corresponding to λ. Let μ be the dual partition of λ and u a unipotent element of $GL_n(\mathbb{C})$ whose Jordan blocks are given by the partition μ. Denote by F_λ a maximal reductive subgroup of the centralizer of u in $GL_n(\mathbb{C})$. According to Theorem 2.3.2 (c) and Lemma 2.3.1, to prove the conjecture of Lusztig for $J_{\mathbf{c}}$ we only need to prove the following special case of the conjecture.

Conjecture 2.3.3. There is a bijection $\pi : \Gamma_\lambda^{-1} \cap \Gamma_\lambda \to \mathrm{Irr}F_\lambda$ such that
(a) The map $t_w \to \pi(w)$ defines a ring isomorphism from $J_{\Gamma_\lambda^{-1}\cap\Gamma_\lambda}$ to R_{F_λ}.
(b) $\pi(w^{-1}) = \pi(w)^*$ for any $w \in \Gamma_\lambda^{-1} \cap \Gamma_\lambda$. (Recall that $\pi(w)^*$ is the dual of $\pi(w)$.)

We will prove this conjecture in Chapter 8. To define the map π in 2.3.3 (a) we need some properties of antichains. In the following section we give some discussions to chains and antichains.

2.4. Chains and antichains

In Chapetr 5 we will define the map π in Conjecture 2.3.3 by means of r-antichains. In this section we prove some results about chains and antichains which will be used later. In this section w stands for an element of W.

Lemma 2.4.1. *If i and j are in a d-chain (resp. a d-antichain) of w, then any d-antichain (resp. d-chain) of w contains at most one of $i + an, j + bn$ for any given integers a, b.*

Proof. Suppose that $i < j$ are in a d-chain of w. Then $w(i) > w(j)$.

Assume that $i + an$ and $j + bn$ are in a d-antichain of w for some integers a, b. If

$$j + (b - 1)n < i + an < j + bn,$$

then we have $b - a \le 0$ since $i < j$. Thus

$$w(i + an) = w(i) + an > w(j + bn) = w(j) + bn$$

since $w(i) > w(j)$. In this case $i + an$ and $j + bn$ can not be in the same d-antichain of w. If

$$i + an > j + bn > i + (a - 1)n,$$

then $b - a \le -1$ since $i < j$. Then we have

$$w(j + bn) = w(j) + bn < w(i + an - n) = w(i) + (a - 1)n$$

since $w(i) > w(j)$. In this case $i + an$ and $j + bn$ are also not in the same d-antichain of w.

Suppose that $i < j$ are in a d-antichain of w, then $j - n < i$ and $w(j) - n < w(i) < w(j)$. Assume that $i + an$ and $j + bn$ are in a d-chain of w for some integers a, b. If $i + an < j + bn$, then $b - a \geq 0$ since $j - n < i$. Thus

$$w(i + an) = w(i) + an < w(j + bn) = w(j) + bn$$

since $w(i) < w(j)$. In this case $i + an$ and $j + bn$ can not be in the same d-chain of w. If $i + an > j + bn$, then $a - b \geq 1$ since $i < j$. Then we have

$$w(j + bn) = w(j) + bn < w(i + an) = w(i) + an$$

since $w(i) > w(j) - n$. In this case both $i + an$ and $j + bn$ are also not in the same d-chain of w.

The lemma is proved.

Corollary 2.4.2. *If $w(i)$ and $w(j)$ are in an r-chain (resp. r-antichain) of w, then any d-antichain (resp. d-chain) of w^{-1} contains at most one of $w(i) + an, w(j) + bn$ for any given integers a, b.*

Proof. Since $w(i)$ and $w(j)$ are in a d-chain (resp. d-antichain) of w^{-1}, the assertion follows from Lemma 2.4.1.

Lemma 2.4.3. *If $j_1 < j_2 < \cdots < j_k$ is a d-antichain of w of length k, then $j_k - n < j_1 < j_2 < \cdots < j_{k-1}$ is also a d-antichain of length k. (We shall say that*

$$j_1 < j_2 < \cdots < j_k$$

and

$$j_i - an < j_{i+1} - an < \cdots < j_k - an < j_1 - (a-1)n < \cdots < j_{i-1} - (a-1)n$$

are **equivalent antichains** *for any integer a and $1 \leq i \leq k$.)*

Proof. Since

$$w(j_k - n) = w(j_k) - n > w(j_{k-1} - n) = w(j_{k-1}) - n$$

and

$$j_k - n > j_{k-1} - n,$$

we see that the lemma is true.

Proposition 2.4.4. *Let $w \in W$ and $\mu = (\mu_1, ..., \mu_{r'})$ be the dual partition of $\lambda(w)$. Given any $1 \leq k \leq \lambda_1$ and any consecutive n integers $Y = \{a_1, a_1 + 1 ..., a_1 + n - 1\}$, we can find a d-antichain family set of w of index k that is included in Y and has cardinality $\mu_1 + \cdots + \mu_k$.*

Proof. By definition we have a d-antichain family Z of w of index k that has cardinality $\mu_1 + \cdots + \mu_k$. Using Lemma 2.4.3 we see that any d-antichain in Z is equivalent to a d-antichain of w in Y. Noting that equivalent d-antichains have the same congruence classes modulo n, we see that the required d-antichain family set exists. The proposition is proved.

Of course this assertion is false for chain family set. For instance, if $w(i) = n - i + 1 + 2(i-1)n$ for $1 \leq i \leq n$, then $\lambda(w) = (n)$ since $n < n - 1 + n < n - 2 + 2n < \cdots < 1 + (n-1)n$ is a d-chain of w of length n. But $\{1, 2, ..., n\}$ is not a d-chain of w.

Let $w \in W$ and $\lambda(w) = (\lambda_1, ..., \lambda_r)$. Assume that the dual partition of $\lambda(w)$ is $\mu(w) = (\mu_1, ..., \mu_{r'})$. We call a subset Y of $\mathbb{Z}$ a **complete d-chain (resp. d-antichain) set** of w if
(1) Y contains n numbers and any two numbers in Y are noncongruent modulo n,
(2) Y is a disjoint union of r d-chains (resp. r' d-antichains) of w of length $\lambda_1, ..., \lambda_r$ (resp. $\mu_1, ..., \mu_{r'}$) respectively.

Similarly we define **complete r-chain sets** and **complete r-antichain sets** of w.

Definition 2.4.5. Let $w \in W$ and $\mu = (\mu_1, ..., \mu_{r'})$ the dual partition of $\lambda(w)$. We say that $\{A_1, ..., A_k\}$ is a **complete d-antichain family** of w if the following four conditions are satisfied:
(1) all A_i are d-antichains of w,
(2) $k = r'$,
(3) the cardinality of A_i is μ_i for all i,
(4) the union of all A_i is $\{1, 2, ..., n\}$. (Necessarily $A_i \cap A_j = \emptyset$ if $i \neq j$.)

We say that $\{w(A_1), ..., w(A_{r'})\}$ is a **complete r-antichain family** of w if $\{A_1, ..., A_{r'}\}$ is a complete d-antichain family of w.

2.4.6. Example: Let $\lambda = (\lambda_1, ..., \lambda_r)$ be a partition of n. Then w_λ has a complete d-antichain family. In fact, for given $1 \leq i \leq \lambda_1$, let $1 \leq h_i \leq r$ be such that $i \leq \lambda_{h_i}$ but $i \geq \lambda_{h_i+1}$ (we set $\lambda_{r+1} = 0$). Then define

$$A_i = \{i, \lambda_1 + i, \lambda_1 + \lambda_2 + i, ..., \lambda_1 + \cdots + \lambda_{h_i - 1} + i\}.$$

It is easy to see that $A_1, ..., A_{\lambda_1}$ form a complete d-antichain family of w_λ.

In general, an element in W does not have a complete d-antichain family. For example, let $n = 6$ and $(w(1), w(2), w(3), w(4), w(5), w(6)) = (6, 3, 10, 7, 8, 11)$. Then $w \in \Gamma_\lambda$, here $\lambda = (2, 2, 1, 1)$, and w does not have any complete d-antichain family. (I am grateful to the referee for providing a similar example.)

2.5. Star operations for W

In this section we give a result (see Lemma 2.5.2) about the star operation. The following result is a partial generalization of [S, Corollary 4.2.3].

Lemma 2.5.1. *Let $s \in S$ be a simple reflection and $w \in W$. Suppose $a < b$ and $w(a) > w(b)$. Then $sw(a) > sw(b)$ if $sw \geq w$.*

Proof. Write

$$a = a_1 + a_2 n \quad \text{and} \quad b = b_1 + b_2 n,$$

where $1 \leq a_1, b_1 \leq n$ and $a_2, b_2 \in \mathbb{Z}$. Set

$$w(i) = p_i + r_i n, \quad \text{where } 1 \leq p_i \leq n, \quad r_i \in \mathbb{Z}.$$

Assume that $s = s_k$ for some $1 \leq k \leq n-1$. Choose $1 \leq j_k, j_{k+1} \leq n$ such that

$$w(j_k) = k + \xi n, \quad w(j_{k+1}) = k + 1 + \zeta n, \quad \xi, \zeta \in \mathbb{Z}.$$

If $i \neq j_k, j_{k+1}$, then $sw(i) = w(i)$ and $p_i \neq k, k+1$. Thus $(p_i - k)(p_i - k - 1) > 0$. Therefore if $1 \leq i < j \leq n$ and either i or j is not in $\{j_k, j_{k+1}\}$, then

$$\left| \left[\frac{w(j) - w(i)}{n} \right] \right| = \left| \left[\frac{sw(j) - sw(i)}{n} \right] \right|.$$

Now $sw(j_k) = k + 1 + \xi n$ and $sw(j_{k+1}) = k + \zeta n$. If $j_k > j_{k+1}$, we must have $\xi \geq \zeta + 1$ since $sw \geq w$. If $j_k < j_{k+1}$, we have $\xi \geq \zeta$ for the same reason.

If neither a_1 nor b_1 is contained in $\{j_k, j_{k+1}\}$, then we have $sw(a) = w(a) > w(b) = sw(b)$. If a_1 is either j_k or j_{k+1} and b_1 is not in $\{j_k, j_{k+1}\}$, then $sw(a)$ is either equal to

$$k + 1 + \xi n + a_2 n = w(a) + 1$$

or

$$k + \zeta n + a_2 n = w(a) - 1,$$

and

$$sw(b) = w(b) = p_{b_1} + r_{b_1} n + b_2 n.$$

When $sw(a) = w(a) - 1$, we must have $\zeta + a_2 > r_{b_1} + b_2$ or $\zeta + a_2 = r_{b_1} + b_2$ and $k + 1 > p_{b_1}$ since $w(a) > w(b)$. But $p_{b_1} \neq k, k+1$, so we have $sw(a) > sw(b)$. Similarly we see $sw(a) > sw(b)$ if $a_1 \neq j_k, j_{k+1}$ and b_1 is either j_k or j_{k+1}.

Now suppose $\{a_1, b_1\} = \{j_k, j_{k+1}\}$. If $a_1 = j_k < j_{k+1} = b_1$, then

$$sw(a) = w(a) + 1 > w(b) - 1 = sw(b).$$

If $a_1 = j_{k+1} < j_k = b_1$ then

$$sw(a) = k + \zeta n + a_2 n = w(a) - 1$$

and

$$sw(b) = k + 1 + \xi n + b_2 n = w(b) + 1.$$

we must have $\xi \geq \zeta + 1$ since $sw \geq w$. Then $\xi + b_2 > \zeta + a_2$ since $b > a$, but this is impossible since $w(a) > w(b)$. Thus we proved the lemma if $s = s_k, k = 1, 2, ..., n-1$.

Assume that $s = s_0$. Choose $1 \leq j_1, j_n \leq n$ such that

$$w(j_1) = 1 + \xi n,\ w(j_n) = n + \zeta n, \quad \xi, \zeta \in \mathbb{Z}.$$

If i is not in $\{j_1, j_n\}$, then $sw(i) = w(i)$ and $p_i \neq 1, n$. Thus $(p_i - 1)(p_i - n) < 0$. Therefore if $1 \leq i < j \leq n$ and $\{i, j\} \neq \{j_1, j_n\}$, then

$$\left| \left[\frac{w(j) - w(i)}{n} \right] \right| = \left| \left[\frac{sw(j) - sw(i)}{n} \right] \right|.$$

Note that $sw(j_1) = \xi n$ and $sw(j_n) = n + 1 + \zeta n$. If $j_1 > j_n$ we must have $\xi \leq \zeta + 1$ since $sw \geq w$. If $j_1 < j_n$, we have $\xi \leq \zeta$ for the same reason.

If neither a_1 nor b_1 is contained in $\{j_1, j_n\}$, then we have $sw(a) = w(a) > w(b) = sw(b)$. If a_1 is either j_1 or j_n and $b \neq j_1, j_n$, then $sw(a)$ is either

$$\xi n + a_2 n = w(a) - 1$$

or

$$n + 1 + \zeta n + a_2 n = w(a) + 1,$$

and

$$sw(b) = w(b) = p_{b_1} + r_{b_1} n + b_2 n.$$

We must have $\xi + a_2 > r_{b_1} + b_2$ if $sw(a) = w(a) - 1$ since $w(a) > w(b)$. But $p_{b_1} \neq 1, n$, so we have $sw(a) > sw(b)$. Similarly we see $sw(a) > sw(b)$ if $a_1 \neq j_1, j_n$ and b_1 is either j_1 or j_n.

Now suppose $\{a_1, b_1\} = \{j_1, j_n\}$. If $a_1 = j_n < j_1 = b_1$, then $sw(a) = w(a) + 1 > w(b) - 1 = sw(b)$. If $a_1 = j_1 < j_n = b_1$ we must have $\xi + a_2 > \zeta + b_2$ since $w(a) > w(b)$, but this is impossible since $\xi \leq \zeta$ for the reason of $sw \geq w$ and $a_2 \leq b_2$.

The lemma is proved.

Lemma 2.5.2. *Let $i \in \mathbb{Z}$ and $* = \{s_i, s_{i+1}\}$. We have*
(a) *Suppose that $w(i)$ is between $w(i+1)$ and $w(i+2)$, then w is in $D_R(s_i, s_{i+1})$. Moreover,*

$$w^*(a) = \begin{cases} w(a), & \text{if } a \not\equiv i+1, i+2 (\text{mod } n), \\ w(i+2), & \text{if } a = i+1, \\ w(i+1), & \text{if } a = i+2. \end{cases}$$

(b) *Suppose that $w(i+2)$ is between $w(i)$ and $w(i+1)$, then w is in $D_R(s_i, s_{i+1})$. Moreover,*

$$w^*(a) = \begin{cases} w(a), & \text{if } a \not\equiv i, i+1 (\text{mod } n), \\ w(i+1), & \text{if } a = i, \\ w(i), & \text{if } a = i+1. \end{cases}$$

Proof. (a) If $w(i+1) < w(i) < w(i+2)$, using 2.1.3 (f) and Lemma 2.5.1 repeatedly we see $w = us_i$ for some u with $us_i \geq u$ and $us_{i+1} \geq u$. If $w(i+1) > w(i) > w(i+2)$, using 2.1.3 (f) and Lemma 2.5.1 repeatedly we see $w = us_i s_{i+1}$ for some u with $us_i \geq u$ and $us_{i+1} \geq u$. Therefore $w \in D_R(s_i, s_{i+1})$. In both cases we have $w^* = ws_{i+1}$ so that $w^*(a) = w(a)$ if $a \not\equiv i+1, i+2 (\text{mod } n)$, and $w^*(i+1) = w(i+2)$, $w^*(i+2) = w(i+1)$.

The proof of (b) is similar. The lemma is proved.

CHAPTER 3

Canonical Left Cells

In section 2.3 we proved that $J_{\Gamma\cap\Gamma^{-1}}$ is commutative for any left cell Γ of the extended affine Weyl group W associated with $GL_n(\mathbb{C})$, using the fact that it is true for canonical left cells. In this chapter we give some discussion to canonical left cells in W. Although we do not really need the results here for our main purpose, the discussion here maybe is helpful for understanding other types. In other types the based ring $J_{\Gamma\cap\Gamma^{-1}}$ is not commutative in general, one may see this from [X3, Chapter 11], but it is always commutative when Γ is a canonical left cell (see [LX]). Lusztig conjectured that $J_{\Gamma\cap\Gamma^{-1}} \simeq R_{F_{\mathbf{c}}}$ if Γ is a canonical left cell in a two-sided cell $\mathbf{c}$ of an extended affine Weyl group (see [L8]). If this is true, maybe $J_{\Gamma\cap\Gamma^{-1}}$ is a key to understand $J_{\mathbf{c}}$. One more reason to consider canonical left cells is that the intersection $\Gamma\cap\Gamma^{-1}$ has a good presentation if Γ is a canonical left cell (see [LX]). This fact maybe can be used to prove that the bijection between the set of two-sided cells of an extended affine Weyl group and the set of unipotent classes in the corresponding algebraic group (see [L8]) preserves partial orders.

In section 3.1 we recall some reduced expressions for fundamental weights, which can be found in [L2]. In section 3.2 we determine the right cell containing a given dominant weight. In section 3.3 we discuss the shortest elements m_x in the double cosets W_0xW_0 for all dominant weights x. It is not so easy to describe the partition associated with m_x. In section 3.4 we describe the distinguished involution in a canonical left cell of W.

3.1. The dominant weights

Let W be as in Chapter 2, that is, W is the extended affine Weyl group associated with $GL_n(\mathbb{C})$. Define $x_i = \tau_1\tau_2\cdots\tau_i$. Recall that X is the subgroup of W generated by all τ_i. Then the set $X^+ = \{x \in X \mid l(w_0x) = l(w_0) + l(x)\}$ of dominant weights in W consists of the elements $x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}$, $a_1, a_2, ..., a_{n-1} \in \mathbb{N}$ and $a_n \in \mathbb{Z}$. For further investigation we need some reduced expression for x_i, see [L2],

$$\begin{aligned}
x_1 &= \omega s_{n-1}s_{n-2}\cdots s_2s_1\\
x_2 &= \omega^2(s_{n-2}s_{n-3}\cdots s_2s_1)(s_{n-1}s_{n-2}\cdots s_2)\\
&\quad\quad\quad\quad ...\\
x_i &= \omega^i(s_{n-i}s_{n-i-1}\cdots s_1)(s_{n-i+1}s_{n-i}\cdots s_2)\cdots(s_{n-1}s_{n-2}\cdots s_i)\\
&\quad\quad\quad\quad ...\\
x_{n-2} &= \omega^{n-2}(s_2s_1)(s_3s_2)\cdots(s_{n-1}s_{n-2})\\
x_{n-1} &= \omega^{n-1}s_1s_2\cdots s_{n-2}s_{n-1}\\
x_n &= \omega^n.
\end{aligned}$$

In particular, the length of x_i is $(n-i)i$. From the reduced expressions we see

(a) $l(xy) = l(x) + l(y)$ if x and y are in X^+.

(b) $x_i s_i \leq x_i$ for $1 \leq i \leq n-1$ and $s_j x_i = x_i s_j \geq x_i$ if $i \neq j$.

(c) $l(x_i s_i s_{i+1} \cdots s_{n-1}) = l(x_i) - (n-i)$ for $1 \leq i \leq n-1$.

3.2. The right cell containing $x \in X^+$

In this section we describe the partition associated with a dominant weight x in W. This is equivalent to determine the right cell containing x. Let $\Gamma_{\mathbf{c}}$ be the unique left cell contained in a two-sided cell $\mathbf{c}$ of W with $R(\Gamma_{\mathbf{c}}) \subseteq \{s_0\}$. For convenience we set $\Phi_{\mathbf{c}} = \Gamma_{\mathbf{c}}^{-1}$. Obviously we have

(a) If x is a dominant weight in W then x is contained in some $\Phi_{\mathbf{c}}$.

Let λ be a partition of n and $\mu = (\mu_1, \mu_2, ..., \mu_{r'})$ be the dual partition of λ. Let $\mathbf{c}$ be the two-sided cell corresponding to λ.

Lemma 3.2.1. *Keep the notation above. Let $\nu_1, ..., \nu_{r'}$ be a permutation of $\mu_1, ..., \mu_{r'}$. Set $x_\nu = x_{\nu_1} x_{\nu_1+\nu_2} \cdots x_{\nu_1+\cdots+\nu_{r'}}$. Then $x_\nu \in \Phi_{\mathbf{c}}$.*

Proof. Set $\nu_{10} = 0$ and $\nu_{1i} = \nu_1 + \cdots + \nu_i$ for $i = 1, 2, ..., r'$. Define $V_i = \{\nu_{1,i-1} + 1, \nu_{1,i-1}+2, ..., \nu_{1i}\}$ for $i = 1, 2, ..., r'$. For any $a \in V_i$ we have $x_\nu(a) = a + (r' - i + 1)n$. Thus V_i is a d-antichain of x_ν.

If $a \in V_i$, $b \in V_j$ and $i < j$, then $x_\nu(a) > x_\nu(b)$. Using Lemma 2.4.2 we see that $a + kn$ and $b + ln$ can not be in the same d-antichain of x_ν for any integers k, l. Therefore $V_1, V_2, ..., V_{r'}$ must form a complete d-antichain family of x_ν (see 2.4.5 for definition) and the corresponding partition is μ. So x_ν is in $\Phi_{\mathbf{c}}$.

Lemma 3.2.2. *Let I be a subset of $N = \{1, 2, ..., n-1\}$. Then*
(a) *The elements $\prod_{i \in I} x_i^{a_i}$ and $x_I = \prod_{i \in I} x_i$ are contained in the same right cell of W if all $a_i \geq 1$.*
(b) *The elements $x_n^k \prod_{i \in I} x_i^{a_i}$($a_i \geq 1, k \in \mathbb{Z}$) are contained in the right cell containing x_I.*

Proof. We can find a partition $\mu = (\mu_1, ..., \mu_{r'})$ of n and a permutation $\nu_1, ..., \nu_{r'}$ of $\mu_1, ..., \mu_{r'}$ such that $x_I x_n = x_\nu$ (see Lemma 3.2.1 for definition). Let $V_1, ..., V_{r'}$ be as in the proof of Lemma 3.2.1. Then we see that $V_1, ..., V_{r'}$ form a complete d-antichain family for both $\prod_{i \in I} x_i^{a_i}$ and x_I if all $a_i \geq 1$. Therefore (a) is true. (b) follows from (a) since x_n is in the center of W. The lemma is proved.

The two lemmas show that it is easy to determine the right cell containing a given dominant weight.

Examples:
(1) $x_1 x_2 \cdots x_{n-1}$ is contained in the lowest two-sided cell of W.
(2) x_i and x_{n-i} are contained in the same two-sided cell (we set $x_0 = e$, the neutral element of W). The partition corresponding to the two-sided cell is (2,2,...,2,1,...,1), where 2 appears $\min\{i, n-i\}$ times.

3.3. The elements m_x

It is well known that W is the union of the double cosets W_0xW_0 ($x \in X^+$). For the double coset W_0xW_0 we have a unique element m_x of minimal length. The elements m_x ($x \in X^+$) are interesting since we have (see [LX])

(a) $\bigcup \Gamma_{\mathbf{c}} \cap \Gamma_{\mathbf{c}}^{-1} = \{m_x \mid x \in X^+\}$, where $\mathbf{c}$ runs through the set of two-sided cells of W.

(b) $m_x \neq m_y$ if x, y are in X^+ and $x \neq y$.

(c) Let $w \in W$. Then $L(w) = R(w) \subseteq \{s_0\}$ if and only if $w = m_x$ for some $x \in X^+$.

Since $x_n = \omega^n$ is in the center of W, we have

(d) $m_{xx_n^a} = x_n^a m_x = \omega^{an} m_x$ if $x \in X^+$ and a is an integer.

In this section we work out an explicit form for m_x. Unfortunately the author is unable to describe the left cell containing m_x. Fix a subset I of $N = \{1, 2, ..., n-1\}$. Recall $x_I = \prod_{i \in I} x_i$. We set $m_I = m_{x_I}$.

Lemma 3.3.1. *Let I be a subset of N and $x = \prod_{i \in I} x_i^{a_i}$. Then $m_x = x x_I^{-1} m_I$ if $a_i \geq 1$ for all i.*

Proof. Let $y = x^{-1}$, $y_I = x_I^{-1}$, and $m_I' = m_I^{-1}$. Then $l(s_i m_I') = 1 + l(m_I')$ for $1 \leq i \leq n-1$. Obviously we have

$$m_I' y_I^{-1} y s_i \geq m_I' y_I^{-1} y$$

for $i = 1, 2, ..., n-1$. Thus we only need to show that $l(s_i m_I' y_I^{-1} y) = 1 + l(m_I' y_I^{-1} y)$ for $1 \leq i \leq n-1$. Let $w \in W_0$ be such that $y_I = w^{-1} m_I'$. Recall that R is the root system of W. Let R^+ be the set of positive roots in R and $R^- = -R^+$. We have (see [IM])

$$l(s_i w y_I) = \sum_{\substack{s_i w(\alpha) \in R^- \\ \alpha \in R^+}} |\langle y_I, \alpha^\vee \rangle + 1| + \sum_{\substack{s_i w(\alpha) \in R^+ \\ \alpha \in R^+}} |\langle y_I, \alpha^\vee \rangle|,$$

$$l(w y_I) = \sum_{\substack{w(\alpha) \in R^- \\ \alpha \in R^+}} |\langle y_I, \alpha^\vee \rangle + 1| + \sum_{\substack{w(\alpha) \in R^+ \\ \alpha \in R^+}} |\langle y_I, \alpha^\vee \rangle|.$$

If $s_i w \geq w$, we can find a unique $\beta \in R^+$ such that $s_i w(\beta) \in R^-$ and $w(\beta) \in R^+$. Then by the formulas above we have $|\langle y_I, \beta^\vee \rangle + 1| > |\langle y_I, \beta^\vee \rangle|$ since $l(s_i w y_I) > l(w y_I)$. Since $\langle y_I, \beta^\vee \rangle \leq 0$, we necessarily have $\langle y_I, \beta^\vee \rangle = 0$. This implies that $\langle y, \beta^\vee \rangle = 0$. By the length formulas for $l(s_i m_I' y_I^{-1} y) = l(s_i w y)$ and for $l(m_I' y_I^{-1} y) = l(wy)$ we see $l(s_i m_I' y_I^{-1} y) = 1 + l(m_I' y_I^{-1} y)$.

If $s_i w \leq w$, we can find a unique $\beta \in R^+$ such that $s_i w(\beta) \in R^+$ and $w(\beta) \in R^-$. Then by the formulas above we have $|\langle y_I, \beta^\vee \rangle + 1| < |\langle y_I, \beta^\vee \rangle|$ since $l(s_i w y_I) > l(w y_I)$. Since $\langle y_I, \beta^\vee \rangle \leq 0$, we necessarily have $\langle y_I, \beta^\vee \rangle < 0$. This implies that $\langle y, \beta^\vee \rangle < 0$. By the length formulas for $l(s_i m_I' y_I^{-1} y)$ and for $l(m_I' y_I^{-1} y)$ we see $l(s_i m_I' y_I^{-1} y) = 1 + l(m_I' y_I^{-1} y)$.

The lemma is proved.

By the lemma above, to get an explicit form for m_x we only need to get an explicit form for m_I. We need some notation. For $1 \le i \le j \le n-1$ we define $s_{ij} = s_i s_{i+1} \cdots s_j$.

Lemma 3.3.2. *Suppose $I = \{a_k > a_{k-1} > \cdots > a_1\}$ is a subset of N. Then we have*

$$m_I = x_I x_{a_1}^{-1} w_k w_{k-1} \cdots w_2 \omega^{a_1},$$

where $w_i = s_{a_i,n-1} s_{a_i-1,n-2} s_{a_i-2,n-3} \ldots s_{a_{i-1}+1,n+a_{i-1}-a_i}$ for $i = k, k-1, ..., 2$.

Proof. It is easy to see that $w = x_{a_1}^{-1} w_k w_{k-1} \cdots w_2 \omega^{a_1}$ is in W_0. Therefore $z = x_I w$ is in the double coset $W_0 x_I W_0$. Using the reduced expressions in 3.1 and using 3.1 (a) and 3.1 (b), we see $l(z) = l(x_I) - l(w)$. Thus $L(z) \subseteq L(x_I) \subseteq \{s_0\}$. We also need show that $R(z) \subseteq \{s_0\}$. We can write down explicitly the action of z on $\{1, 2, ..., n\}$, which is,

$$\begin{array}{rl}
1 & \to a_k + 1 \\
2 & \to a_k + 2 \\
 & \ldots \\
n - a_k & \to n \\
n - a_k + 1 & \to a_{k-1} + 1 + n \\
n - a_k + 2 & \to a_{k-1} + 2 + n \\
 & \ldots \\
n - a_{k-1} & \to a_k + n \\
 & \ldots \\
n - a_i + 1 & \to a_{i-1} + 1 + \xi_i n \\
n - a_i + 2 & \to a_{i-1} + 2 + \xi_i n \\
 & \ldots \\
n - a_{i-1} & \to a_i + \xi_i n \\
 & \ldots \\
n - a_1 + 1 & \to 1 + \xi_1 n \\
n - a_1 + 2 & \to 2 + \xi_1 n \\
 & \ldots \\
n & \to a_1 + \xi_1 n
\end{array}$$

where $\xi_i = k - i + 1$. In other words, we have $z(n - a_i + j) = a_{i-1} + j + \xi_i n$ for $1 \le i \le k$, $1 \le j \le a_i - a_{i-1}$ (we set $a_0 = 0$ and $a_{k+1} = n$). Thus $z(i) > z(j)$ if $n \ge i > j \ge 1$. Using Lemma 2.5.1 we see $R(z) \subseteq \{s_0\}$.

The lemma is proved.

Examples:

(1) $m_N = x_N w_0$.

(2) $m_{x_i} = \omega^i$ for all i. Thus $m_{x_i^k} = x_i^{k-1} \omega^i$ if $k \ge 1$.

(3) Let $n \ge q \ge p \ge 1$. If $I = \{q, q-1, ..., p+1, p\}$, then

$$m_I = x_q x_{q-1} \cdots x_{p+1} s_{q,n-1} s_{q-1,n-1} \cdots s_{p+1,n-1} \omega^p.$$

Lemma 3.3.3. *Let I be a subset of $\{1, 2, ..., n-1\}$. Suppose that $x = \prod_{i \in I} x_i^{a_i}$ and $a_i \ge 2$ for all i in I. Then $m_x \underset{R}{\sim} x_I$.*

Proof. We have $m_x = xx_I^{-1}m_I$. Now $x \underset{R}{\sim} xx_I^{-1} \underset{R}{\sim} x_I$ since $a_i \geq 2$ for all i, we necessarily have $m_x \underset{R}{\sim} x_I$ since $x \underset{R}{\leq} m_x \underset{R}{\leq} xx_I^{-1}$. The lemma is proved.

Let μ be the dual partition of λ. By Lemma 3.3.3 we know $m_{x_\mu^2} = x_\mu m_{x_\mu}$ is contained in the two-sided cell $\mathbf{c}$ corresponding to λ. Using Lemma 2.5.2 and the table in the proof of Lemma 3.3.2, it is easy to find the sequence of right star operations and i for passing $\Gamma_{\mathbf{c}}$ to $\omega^i\Gamma_\lambda\omega^{-i}$.

3.4. The distinguished involutions

Now we determine the distinguished involution in a canonical left cell. The involutions look complicated.

Lemma 3.4.1. *Suppose that m_x ($x \in X^+$) is a distinguished involution and*

$$x = x_1^{a_1}x_2^{a_2}\cdots x_{n-2}^{a_{n-2}}x_{n-1}^{a_{n-1}}x_n^{a_n}.$$

Then

(a) $a_i = a_{n-i}$ *for all* $1 \leq i \leq n-1$.

(b) $a_1 + 2a_2 + \cdots + (n-1)a_{n-1} + na_n = 0$.

Proof. (a) According to the proof of [LX, Theorem 3.5], we have $m_x^{-1} = m_{w_0x^{-1}w_0}$. The required assertion then follows from $m_x^{-1} = m_x$, $w_0x^{-1}w_0 = x_1^{a_{n-1}}x_2^{a_{n-2}}\cdots x_{n-2}^{a_2}x_{n-1}^{a_1}x_n^{a_n}$ and 3.3 (b).

(b) Since $m_x \in W'$, we have $x \in W'$. Using the reduced expressions in 3.1 we see that (b) is true. Recall that W' is the subgroup of W generated by all simple reflections $s_0, s_1, ..., s_{n-1}$, see 2.1.3 (c).

The lemma is proved.

Theorem 3.4.2. *The following elements are all distinguished involutions contained in canonical left cells.*

(a) $$\omega^{-(a-1)n}m_{x_{i_1}x_{i_2}\cdots x_{i_{a-1}}x_{n-i_{a-1}}\cdots x_{n-i_2}x_{n-i_1}},$$

where $1 \leq i_1 < i_2 \cdots < i_{a-1}$ *satisfies that* $i_{h+1} - i_h - (i_{h-1} - i_{h-2}) \geq 0$ *for* $h = 1, ..., a-2$, *(we set* $i_0 = i_{-1} = 0$,*) and* $n - 2i_{a-1} - (i_{a-2} - i_{a-3}) \geq 0$. *The associated partition is*

$$(a, ..., a, a-1, ..., a-1, a-2, ..., a-2, ..., a-h, ..., a-h, ..., 2, ..., 2, 1, ..., 1),$$

where a *appears* i_1 *times,* $a-h$ *appears* $i_{h+1} - i_h - (i_{h-1} - i_{h-2})$ *times for* $h = 1, 2, ..., a-2$, *and 1 appears* $n - 2i_{a-1} - (i_{a-2} - i_{a-3})$ *times.*

(b) $$\omega^{-2pn}m_{x_{i_1}^2x_{i_2}^2\cdots x_{i_p}^2x_{n-i_p}^2\cdots x_{n-i_2}^2x_{n-i_1}^2},$$

where $1 \leq i_1 < i_2 \cdots < i_p$ *satisfies that* $i_{h+1} - i_h - (i_h - i_{h-1}) \geq 0$ *for* $h = 1, ..., p-1$, *(we set* $i_0 = i_{-1} = 0$,*) and* $n - 2i_p - (i_p - i_{p-1}) \geq 0$. *The associated partition is*

$$(a, ..., a, a-2, ..., a-2, ..., a-2h, ..., a-2h, ..., 3, ..., 3, 1, ..., 1),$$

where a *appears* i_1 *times,* $a-2h$ *appears* $i_{h+1} - i_h - (i_h - i_{h-1})$ *times for* $h = 1, 2, ..., p-1$, *and 1 appears* $n - 2i_p - (i_p - i_{p-1})$ *times, where* $a = 2p+1$.

(c) $$\omega^{-(p+r)n}m_{x_{i_1}^2x_{i_2}^2\cdots x_{i_p}^2x_{i_{p+1}}\cdots x_{i_r}x_{n-i_r}\cdots x_{n-i_{p+1}}x_{n-i_p}^2\cdots x_{n-i_2}^2x_{n-i_1}^2},$$

where $1 \le i_1 < i_2 \cdots < i_p < i_{p+1} < \cdots < i_r$ *satisfies that* $i_{h+1} - i_h - (i_h - i_{h-1}) \ge 0$ *for* $h = 1, ..., p$, *(we set* $i_0 = i_{-1} = 0$,*) and* $i_{h+1} - i_h - (i_{h-1} - i_{h-2}) \ge 0$ *for* $p + 2 \le h \le r - 1$, *and* $n - 2i_r - (i_{r-1} - i_{r-2}) \ge 0$ *if* $r \ge p + 2$, $n - 2i_{p+1} \ge 0$ *if* $r = p + 1$. *The associated partition is*

$$(a, ..., a, a-2, ..., a-2p, ..., a-2p, a-2p-1, ..., a-2p-1, ..., 2, ..., 2, 1, ..., 1),$$

where a *appears* i_1 *times,* $a - 2h$ *appears* $i_{h+1} - i_h - (i_h - i_{h-1})$ *times for* $h = 1, 2, ..., p$, $a - 2p - 1$ *appears* $i_{p+2} - i_{p+1}$ *times if* $r \ge p + 2$, $a - 2p - h$ *appears* $i_{p+h+1} - i_{p+h} - (i_{p+h-1} - i_{p+h-2})$ *times for* $h = 2, 3, ..., a - 2p - 2$, *1 appears* $n - 2i_r - (i_{r-1} - i_{r-2})$ *times if* $r \ge p + 2$ *and appears* $n - 2i_{p+1}$ *times if* $r = p + 1$, *where* $a = p + r + 1$.

Proof. (a) According to the proof of 3.3.2, the action of

$$m = \omega^{-(a-1)n} m_{x_{i_1} x_{i_2} \cdots x_{i_{a-1}} x_{n-i_{a-1}} \cdots x_{n-i_2} x_{n-i_1}}$$

on $\{1, 2, ..., n\}$ is given by the following table,

$$\begin{array}{rl}
1 & \to n - i_1 + 1 - (a-1)n \\
2 & \to n - i_1 + 2 - (a-1)n \\
 & \cdots \\
i_1 & \to n - (a-1)n \\
i_1 + 1 & \to n - i_2 + 1 - (a-2)n \\
i_1 + 2 & \to n - i_2 + 2 - (a-2)n \\
\cdots & \\
i_2 & \to n - i_1 - (a-2)n \\
 & \cdots \\
i_j + 1 & \to n - i_{j+1} + 1 - (a-1-j)n \\
i_j + 2 & \to n - i_{j+1} + 2 - (a-1-j)n \\
 & \cdots \\
i_{j+1} & \to n - i_j - (a-1-j)n \\
 & \cdots \\
i_{a-1} + 1 & \to i_{a-1} + 1 \\
i_{a-1} + 2 & \to i_{a-1} + 2 \\
 & \cdots \\
n - i_{a-1} & \to n - i_{a-1} \\
 & \cdots \\
n - i_j + 1 & \to i_{j-1} + 1 + (a-j)n \\
n - i_j + 2 & \to i_{j-1} + 2 + (a-j)n \\
 & \cdots \\
n - i_{j-1} & \to i_j + (a-j)n \\
 & \cdots \\
n - i_1 + 1 & \to 1 + (a-1)n \\
n - i_1 + 2 & \to 2 + (a-1)n \\
 & \cdots \\
n & \to a_1 + (a-1)n
\end{array}$$

Set $i_0 = 0$. We define

$$\eta_{jk} = \begin{cases} i_{j-1} + k & \text{if } 1 \le j \le a-1,\ 1 \le k \le i_j - i_{j-1} \\ i_{a-1} + k & \text{if } j = a,\ 1 \le k \le n - i_{a-1} \\ n - i_{2a-j} + k & \text{if } a < j \le 2a-1,\ 1 \le k \le i_{2a-j} - i_{2a-1-j}. \end{cases}$$

Then we have

$$m(\eta_{jk}) = \eta_{2a-j,k} + (j-a)n$$

for $1 \le j \le 2a-1$ and all k.

We now define an order among all the pairs (j,k) for which η_{jk} are defined. We say $(j,k) < (j',k')$ if one of the following three cases happens, (1) j is even and j' is odd, (2) both j, j' have the same parity and $k < k'$, (3) both j, j' have the same parity and $j > j'$, $k = k'$. It is easy to check that this defines a total order on the set of such pairs (j,k). Arranging the pairs in the increasing order, then we have a corresponding arrangement $\xi_1, \xi_2, ..., \xi_n$ for all η_{jk} (i.e. if $\xi_l = \eta_{jk}$ and $\xi_{l'} = a_{j'k'}$, then $l < l'$ if and only if $(j,k) < (j',k')$). For a fixed j we denote by c_j the cardinality of the set consisting of all η_{jk}. Set

$$h = \sum_{1 \le j \le 2a-3} (a - 1 - \left[\frac{j}{2}\right]) c_j.$$

We shall use $\psi_j(w)$ for w^* if $w \in D_R(s_j, s_{j+1})$ and $* = \{s_j, s_{j+1}\}$. We also write $\psi_{ij}(w)$ $(i \le j)$ for $\psi_i \psi_{i+1} \cdots \psi_{j-1} \psi_j(w)$ if the latter is defined. Using Lemma 2.5.2 we see that $m_1 = \psi_{i_2+1,n-1}(m)$ is well defined. Moreover $l(m_1) = l(m) - (n - i_2 - 1)$ and $l(\psi_{i_2}(m_1)) = l(m_1) + 1$. Set $m_0 = m$. Suppose for $j = 1, ..., k-1$ $(k \ge 2)$ we have defined $m_j = \psi_{p_j, n+j-2}(m_{j-1})$ with $l(m_j) = l(m_{j-1}) - (n + j - p_j - 1)$ and $l(\psi_{p_j - 1}(m_j)) = l(m_j) + 1$. Then we define $m_k = \psi_{p_k, n+k-2}(m_{k-1})$, here p_k is chosen so that $\psi_{p_k, n+k-2}(m_{k-1})$ is well defined and $l(m_k) = l(m_{k-1}) - (n + k - p_k - 1)$ and $l(\psi_{p_k - 1}(m_k)) = l(m_k) + 1$. By the conditions $i_{h+1} - i_h - (i_h - i_{h-1}) \ge 0$ and $n - 2i_{a-1} - (i_{a-2} - i_{a-3}) \ge 0$, such $p_1, p_2, ..., p_k$ exist. Continuing this process we finally get an element m' whose action on $\{h+1, h+2, ..., h+n\}$ is given by $m'(h+l) = m(\xi_l) + qn$, where $q = (a - 1 - \left[\frac{j}{2}\right])$ if $\xi_l = \eta_{jk}$. Thus we have

$$m'(h+l) = \eta_{2a-j,k} + (j-a)n + (a - 1 - \left[\frac{j}{2}\right])n$$

if $\xi_l = \eta_{jk}$. Therefore we have

$$m'^{-1}(\eta_{jk}) = h + l - (2a - j - 1 - \left[\frac{2a-j}{2}\right])n,$$

where l is defined by $\xi_l = \eta_{2a-j,k}$. Applying the same sequence of right star operations $\psi_{p_1,n-1}, \psi_{p_2,n}, ...$ (in the same order) to m'^{-1} we get an element m'' whose action on $\{h+1, h+2, ..., h+n\}$ is the same as the action of w_I on, here I is a suitable subset of $\{h+1, h+2, ..., h+n\}$, and w_I is the longest element of the subgroup generated by all s_j $(j \in I)$. Moreover, $\lambda(w_I)$ is the partition in (a). Using Prop. 1.4.6 we see that (a) is true.

(b) Using Lemma 3.3.1 and the proof of Lemma 3.3.2 we can write down explicitly the action on $\{1, 2, ..., n\}$ of the element in (b). Then as the proof of (a) we can see that the element in (b) is a distinguished involution and its associated partition is the given one in (b).

The proof for part (c) is similar.

The theorem is proved.

Examples:
The following elements are distinguished involutions, (1) $\omega^{-n}m_{x_ix_{n-i}}$ $(i \leq \frac{n}{2})$, (2) $\omega^{-2n}m_{x_2^2x_{n-2}^2}$, (3) $\omega^{-3n}m_{x_1^2x_ix_{n-i}x_{n-1}^2}$ $(i \leq \frac{n}{2})$, (4) $\omega^{-(n-1)n}$ $\times m_{x_1^2x_2^2\cdots x_{n-1}^2}$. The associated partitions are, (1) (2,...,2,1,...,1), where 2 appears i times, (2) (3,3,1,...,1), (3) (4,2,...,2,1,...,1), where 2 appears $i-2$ times, (4) (n).

CHAPTER 4

The Group F_λ and Its Representation

For later use we discuss the group F_λ and its representations in this chapter. In section 4.1 we give an explicit description for the group F_λ. In section 4.2 we give some facts about the representations of F_λ. For completeness we also supply a few proofs although the facts are well known.

4.1. The group F_λ

Let W be as in Chapter 2, that is, W is the extended affine Weyl group associated with $GL_n(\mathbb{C})$. For each two-sided cell $\mathbf{c}$ of W we have a corresponding partition λ of n. Let $\mu = (\mu_1, \mu_2, ..., \mu_{r'})$ be the dual partition of λ. Let u be a unipotent element in $GL_n(\mathbb{C})$ whose Jordan blocks are determined by the partition μ. Choose $r' \geq j_1 > j_2 > \cdots > j_p \geq 1$ such that
(1) $\mu_{j_1} < \mu_{j_2} < \cdots < \mu_{j_p}$,
(2) for any $1 \leq i \leq r'$ we have $\mu_i = \mu_{j_k}$ for some k.
Let

$$n_k = \#\{i \mid 1 \leq i \leq r' \text{ and } \mu_i = \mu_{j_k}\}.$$

Let $C_G(u)$ be the centralizer of u in $G = GL_n(\mathbb{C})$. Then the maximal reductive subgroup $F_\lambda = F_{\mathbf{c}}$ of $C_G(u)$ is isomorphic to $GL_{n_1}(\mathbb{C}) \times GL_{n_2}(\mathbb{C}) \times \cdots \times GL_{n_p}(\mathbb{C})$. We shall identify F_λ with $GL_{n_1}(\mathbb{C}) \times GL_{n_2}(\mathbb{C}) \times \cdots \times GL_{n_p}(\mathbb{C})$.

Examples:
(1) If $\lambda = (n)$, then $\mu = (1, ..., 1)$. In this case $F_\lambda = GL_n(\mathbb{C})$.
(2) If $\lambda = (1, ..., 1)$, then $\mu = (n)$. In this case $F_\lambda \simeq \mathbb{C}^*$.
(3) If $\lambda = (2, 1, ..., 1)$, then $\mu = (n-1, 1)$. Thus we have $F_\lambda \simeq \mathbb{C}^* \times \mathbb{C}^*$ if $n \geq 3$.

We define $\mathbb{Z}^n_{\text{dom}}$ to be the set $\{(r_1, r_2, ..., r_n) \in \mathbb{Z}^n \mid r_1 \geq r_2 \geq \cdots \geq r_n\}$. The elements in $\mathbb{Z}^n_{\text{dom}}$ will be called dominant elements in $\mathbb{Z}^n$. It is well known that the set $\text{Irr}GL_n(\mathbb{C})$ of isomorphism classes of irreducible rational representations of $GL_n(\mathbb{C})$ is one to one corresponding to $\mathbb{Z}^n_{\text{dom}}$.

Thus the set $\text{Irr}F_\lambda$ of isomorphism classes of irreducible rational representations of F_λ is one to one corresponding to

$$\text{Dom}(F_\lambda) = \mathbb{Z}^{n_1}_{\text{dom}} \times \mathbb{Z}^{n_2}_{\text{dom}} \times \cdots \times \mathbb{Z}^{n_p}_{\text{dom}}.$$

For an element $\varepsilon = (\varepsilon_{11}, ..., \varepsilon_{1n_1}, ..., \varepsilon_{p1}, ..., \varepsilon_{pn_p})$ in $\text{Dom}(F_\lambda)$ we also call ε_{ij} the (i,j)-component of ε.

4.2. The representation ring of F_λ

For any algebraic group G we use R_G for its (rational) representation ring. We are interested in the representation ring R_{F_λ}. Since F_λ is isomorphic to a direct product of the general linear groups $GL_{n_i}(\mathbb{C})$ $(1 \le i \le p)$, we see that R_{F_λ} is isomorphic to the tensor product (over $\mathbb{Z}$) of the representation rings $R_{GL_{n_i}(\mathbb{C})}$ $(1 \le i \le p)$. Thus we are reduced to understand $R = R_{GL_n(\mathbb{C})}$.

The structure of R is known. For our purpose we list some properties of R. We shall identify X (resp. X^+) with $\mathbb{Z}^n$ (resp. $\mathbb{Z}^n_{\mathrm{dom}}$) by identifying τ_i with the element in $\mathbb{Z}^n$ whose ith component is 1 and other components are 0. For any $x \in X^+$, we denote by $V(x)$ an irreducible representation of $GL_n(\mathbb{C})$ with highest weight x.

(a) As a ring R is generated by $V(x_1), ..., V(x_{n-1}),\ V(x_n), V(x_n^{-1})$, see 3.1 for definition of x_i.

(a') As a ring R is generated by $V(x_n^{-1})$ and $V(x_1^a)$ for $a = 1, 2, ..., n$.

(b) R is a free $\mathbb{Z}$-module. The elements $V(x)$ $(x \in X^+)$ form a $\mathbb{Z}$-basis of R, and the elements

$$V(x_1)^{a_1} V(x_2)^{a_2} \cdots V(x_{n-1})^{a_{n-1}} V(x_n)^{a_n},\ a_1, ..., a_{n-1} \in \mathbb{N},\ a_n \in \mathbb{Z},$$

also form a $\mathbb{Z}$-basis of R.

(b') The elements

$$V(x_1)^{a_1} V(x_1^2)^{a_2} \cdots V(x_1^{n-1})^{a_{n-1}} V(x_n)^{a_n},\ a_1, ..., a_{n-1} \in \mathbb{N},\ a_n \in \mathbb{Z},$$

form a $\mathbb{Z}$-basis of R.

(c) For $1 \le i \le n$, the dimension of $V(x_i)$ is $\binom{n}{i}$.

(d) Each weight space of $V(x_i)$ has dimension one and the weight set of $V(x_i)$ consists of all $\tau_{k_1}\tau_{k_2}\cdots\tau_{k_i}$ $(1 \le k_1 < k_2 < \cdots < k_i \le n)$.

(e) Let $x \in X^+$. Then

$$V(x_i) \otimes V(x) \simeq \oplus_\tau V(\tau x),$$

where τ runs through the set of weights of $V(x_i)$ such that $\tau x \in X^+$.

Proof. Let $\delta = \tau_1^{n-1}\tau_2^{n-2}\cdots\tau_{n-1}$. By (d), $x\tau\delta \in X^+$ for any weight τ of $V(x_i)$. Using the tensor product formula in [H, Ex. 24.9] we see that (e) is true.

(f) Let K be a ring. Assume that K is a free $\mathbb{Z}$-module with a basis $\{t_x \mid x \in X^+\}$. If for all i we have $t_{x_i}t_x = \sum_\tau t_{\tau x}$, here τ runs through the set of weights of $V(x_i)$ with $\tau x \in X^+$, then we have a ring isomorphism between R and K defined by $V(x) \to t_x$.

Proof. It follows from (a), (b) and (e).

(g) Let $x \in X^+$ and $a \in \mathbb{N}$. Then

$$V(x_1^a) \otimes V(x) \simeq \oplus_\tau V(\tau x),$$

where τ runs through the set of weights $\tau_1^{a_1}\tau_2^{a_2}\cdots\tau_n^{a_n}$ of $V(x_1^a)$ with $\tau_i^{a_i}\tau_{i+1}^{a_{i+1}}\cdots\tau_n^{a_n}x \in X^+$ for $i = 1, 2, ..., n$. (Note that each weight space of $V(x_1^a)$ has dimension one and the weight set of $V(x_1^a)$ consists of all $\tau_1^{a_1}\tau_2^{a_2}\cdots\tau_n^{a_n}$, $a_1, ..., a_n \in \mathbb{N}$ and $a_1 + a_2 + \cdots + a_n = a$.)

Proof. It follows from Littlewood-Richardson rule. When $a = 2$, we also can see (g) using [H, Ex. 24.9].

Recently Littelmann generalized the rule to arbitrary types, see [Li].

(h) Let K be a ring. Assume that K is a free $\mathbb{Z}$-module with a basis $\{t_x \mid x \in X^+\}$. If for all positive integers a we have $t_{x_1^a} t_x = \sum_\tau t_{\tau x}$, here τ runs through the set of weights $\tau_1^{a_1}\tau_2^{a_2}\cdots\tau_n^{a_n}$ of $V(x_1^a)$ with $\tau_i^{a_i}\tau_{i+1}^{a_{i+1}}\cdots\tau_n^{a_n}x \in X^+$ for $i = 1, 2, ..., n$, then we have a ring isomorphism between R and K defined by $V(x) \to t_x$.

Proof. It follows from (a'), (b') and (f).

CHAPTER 5

A Bijection Between $\Gamma_\lambda \cap \Gamma_\lambda^{-1}$ And IrrF_λ

In this chapter we will establish a bijection between $\Gamma_\lambda \cap \Gamma_\lambda^{-1}$ and the set of dominant weights of F_λ. This bijection gives rise to a bijection between $\Gamma_\lambda \cap \Gamma_\lambda^{-1}$ and IrrF_λ, that is in fact the bijection π in Conjecture 2.3.3. We use some r-antichains of elements in $\Gamma_\lambda \cap \Gamma_\lambda^{-1}$ to define the bijection. To do this we show that each element in $\Gamma_\lambda \cap \Gamma_\lambda^{-1}$ has a complete r-antichain family, see 2.4.5 for definition. The contents of this chapter is as follows. In section 5.1 we show that each element in $\Gamma_\lambda \cap \Gamma_\lambda^{-1}$ has a complete r-antichain family, see Theorem 5.1.12. In section 5.2 we define the map ε from $\Gamma_\lambda \cap \Gamma_\lambda^{-1}$ to the set of dominant weights of F_λ by means of r-antichains of elements in $\Gamma_\lambda \cap \Gamma_\lambda^{-1}$ and give some discussions to the map. The main result of this chapter is Theorem 5.2.6, which says that the map ε is bijective. In section 5.3 we prove the main theorem by describing elements of $\Gamma_\lambda \cap \Gamma_\lambda^{-1}$. In section 5.4 we give some simple properties of elements in $\Gamma_\lambda \cap \Gamma_\lambda^{-1}$. In section 5.5 we give some elements in $\Gamma_\lambda \cap \Gamma_\lambda^{-1}$, most of them correspond to fundamental weights. In next 3 chapters we will show that this bijection provides the isomorphism between $J_{\Gamma_\lambda \cap \Gamma_\lambda^{-1}}$ and the rational representation ring R_{F_λ}.

5.1. r-antichains of elements in $\Gamma_\lambda \cap \Gamma_\lambda^{-1}$

Let λ be a partition of n. Recall that we have defined the element w_λ in §2.2 and Γ_λ is the left cell of W containing w_λ. In this section we show that each element in $\Gamma_\lambda \cap \Gamma_\lambda^{-1}$ has a complete r-antichain family. This is a rather delicate result since it does not hold even for some elements in Γ_λ, see the example in 2.4.6.

Let $\lambda = (\lambda_1, ..., \lambda_r)$ and $\mu = (\mu_1, ..., \mu_{r'})$ be its dual. Let $n_1, ..., n_p$ be as in 4.1. The numbers $n_1, ..., n_p$ can be defined in another way. Choose $0 = r_0 < r_1 < r_2 < \cdots < r_k = r$ such that

$$\lambda_{r_i+1} = \lambda_{r_i+2} = \cdots = \lambda_{r_{i+1}} > \lambda_{r_{i+1}+1} = \cdots = \lambda_{r_{i+2}}$$

for $i = 0, 1, ..., k-2$. Then $k = p$ and $n_i = \lambda_{r_i} - \lambda_{r_{i+1}}$ for all $i = 1, 2, ..., p$ (we understand that $\lambda_{r_{p+1}} = 0$).

Lemma 5.1.1. *Let $w \in \Gamma_\lambda$. Define $e_i = \lambda_1 + \lambda_2 + \cdots + \lambda_i$ for $1 \le i \le r$ and $e_0 = 0$. Then*

(a) $w(e_i + 1) > w(e_i + 2) > \cdots > w(e_i + \lambda_{i+1})$ *for $i = 0, 1, ..., r-1$.*

(b) *Given $0 \le h < i \le r-1$ and $1 \le j \le \lambda_{i+1}$, we have*

$$w(e_i + j) > w(e_h + \lambda_{h+1} - \lambda_{i+1} + j).$$

In particular, we have

$$w(e_i + j) > w(e_{i-1} + \lambda_i - \lambda_{i+1} + j).$$

(c) *$w(e_{i-1}+k+n) > w(e_j+k)$ for $j=i,i+1,...,r-1$ and $1 \le k \le \lambda_{j+1}$.*

Proof. Since $w \in \Gamma_\lambda$, $w = w_1 w_\lambda$ for some w_1 with $l(w_1 w_\lambda) = l(w_1) + l(w_\lambda)$. Using Lemma 2.5.1 we see that (a) is true.

(b) Suppose that the assertion is not true. Then

$$w(e_i+j) < w(e_h + \lambda_{h+1} - \lambda_{i+1} + j).$$

Thus

$$\begin{aligned} &w(1) > w(2) > \cdots > w(\lambda_1),\\ &\ \ ...\\ &w(e_{h-1}+1) > w(e_{h-1}+2) > \cdots > w(e_{h-1}+\lambda_h),\\ &w(e_h+1) > w(e_h+2) > \cdots > w(e_h+\lambda_{h+1}-\lambda_{i+1}+j)\\ &\quad > w(e_i+j) > \cdots > w(e_i+\lambda_{i+1}),\end{aligned}$$

provides an r-chain family set of w of index $h+1$. The cardinality of the set is $e_{h+1}+1$ since the lengths of the $h+1$ r-chains of w in the r-chain family set are $\lambda_1,...,\lambda_h,\lambda_{h+1}+1$ respectively. This contradicts that $\lambda(w)=\lambda$. (b) is proved.

Similarly we prove (c). The lemma is proved.

5.1.2. Given $1 \le j \le r$ and $1 \le k \le \lambda_j$ we set

$$a_{jk} = e_{j-1} + k$$

and

$$\Lambda_j = \{a_{j1},\ a_{j2},\ ...,\ a_{j\lambda_j}\}.$$

We have $w_\lambda(\Lambda_j) = \Lambda_j$ for all $j=1,2,...,r$.

Before going further we give some discussions to complete r-antichian families.

Let w be in $\Gamma_\lambda \cap \Gamma_\lambda^{-1}$ and assume that w has a complete d-antichain family (see 2.4.5 for definition). By the argument for Prop. 2.4.4, we may decompose the set $\{1,2,...,n\}$ into r' d-antichains $A_1, A_2, ..., A_{r'}$ of w whose lengths are $\mu_1,...,\mu_{r'}$ respectively. By Lemma 5.1.1 and Lemma 2.4.1, any intersection $A_i \cap \Lambda_j$ contains at most one element. Thus we have

5.1.2 (a) The d-antichain A_l of w containing $a_{ij} = e_{i-1}+j$ $(1 \le i \le r,\ 1 \le j \le \lambda_i)$ has length $\ge i$.

The d-antichains $A_1, A_2, ..., A_{r'}$ form a complete d-antichain family Z of w and $w(A_1),\ w(A_2), ..., w(A_{r'})$ form a complete r-antichain family of w, see §2.4.5 for definition.

Let Z be a complete r-antichain family of w and $\mu_{j_1} < \mu_{j_2} < \cdots < \mu_{j_p}$ be as in section 4.1. Then we have $\mu_{j_i} = r_i$. Thus Z contains n_i r-antichains of w of length r_i. Let $B_{i1},...,B_{in_i}$ be the r-antichains in Z of length r_i. Let

$$b_{r_i,j} + c_{r_i,j}n > b_{r_i-1,j} + c_{r_i-1,j}n > \cdots > b_{1,j} + c_{1,j}n$$

be elements in B_{ij}, where $1 \le b_{k,j} \le n$ and $c_{k,j} \in \mathbb{Z}$ for all $1 \le k \le r_i$ and $1 \le j \le n_i$.

Lemma 5.1.3. *Let w be in $\Gamma_\lambda \cap \Gamma_\lambda^{-1}$ and assume that w has a complete r-antichain family. Keep the notation in 5.1.2. For $1 \le j \le n_i$, let C_{ij} be the set consisting of all $b_{k,j}$ $(1 \le k \le r_i)$. Then C_{ij} contains exactly one element of Λ_k if $1 \le k \le r_i$ and contains no element of Λ_k if $k > r_i$.*

Proof. Using Corollary 2.4.2, any d-chain of w^{-1} contains at most one element of C_{ij}. But w^{-1} is in Γ_λ, so Λ_h is a d-chain of w^{-1} for $h = 1, ..., r$. Thus $\Lambda_h \cap C_{ij}$ contains at most one element for any h, i, j. Note that C_{ij} contains r_i elements, $n_i = \lambda_{r_i} - \lambda_{r_{i+1}}$ and $h \le r = r_p$. Therefore C_{pj} contains exactly one element of Λ_h for all h and $1 \le j \le n_p$. This forces that each $C_{p-1,j}$ $(1 \le j \le n_{p-1})$ contains exactly one element of Λ_h if $1 \le h \le r_{p-1}$ and contains no element if $h > r_{p-1}$. Inductively we see that C_{ij} contains exactly one element of Λ_k if $1 \le k \le r_i$ and contains no element of Λ_k if $k > r_i$. The lemma is proved.

Lemma 5.1.4. *Let w be in $\Gamma_\lambda \cap \Gamma_\lambda^{-1}$ and assume that w has a complete r-antichain family. Keep the notation in 5.1.2. Given $1 \le i \le p$ and $1 \le j \le n_i$, if $b_{r_i,j}$ is in Λ_k, then*

(a) *$b_{r_i-h,j}$ is in Λ_{k-h} for all $1 \le h < k$, and $b_{r_i-k-h,j}$ is in Λ_{r_i-h} if $0 \le h < r_i - k$.*
(b) *$c_{r_i,j} = \cdots = c_{r_i-k+1,j}$, and $c_{r_i-k,j} = \cdots = c_{1,j} = c_{r_i,j} - 1$.*

Proof. (a) Let b_q be the unique number in $C_{ij} \cap \Lambda_q$ for $1 \le q \le r_i$. Then $b_k = b_{r_i,j}$. We claim that $b_h = b_{r_i-h',j}$ for some $0 \le h' < k$ if $1 \le h \le k$. Otherwise, we have $b_h = b_{r_i-h',j}$ for some $1 \le h \le k$ and $h' \ge k$. Then we can find some $k < q \le r_i$, $1 \le q' < k-1$ such that $b_q = b_{r_i-q',j}$. Since

$$b_{r_i-q',j} = b_q > b_{r_i,j} = b_k > b_{r_i-h',j} = b_h$$

and

$$b_{r_i,j} + c_{r_i,j}n > b_{r_i-q',j} + c_{r_i-q',j}n > b_{r_i-h',j} + c_{r_i-h',j}n,$$

we see that

$$c_{r_i,j} > c_{r_i-q',j} \ge c_{r_i-h'}.$$

Thus

$$b_{r_i,j} + c_{r_i,j}n - b_{r_i-h',j} - c_{r_i-h',j}n > n.$$

This is impossible since B_{ij} is an r-antichain of w. So $b_h = b_{r_i-h',j}$ for some $0 \le h' < k$ if $1 \le h \le k$.

For $1 \le q < q' \le k$, we then have $0 \le h, h' \le k-1$ such that $b_q = b_{r_i-h,j}$ and $b_{q'} = b_{r_i-h',j}$. We claim that $h > h'$. Otherwise, we have $h < h'$. Then $c_{r_i-h,j} > c_{r_i-h',j}$ since $b_q < b_{q'}$ and $b_q + c_{r_i-h,j}n > b_{q'} + c_{r_i-h',j}n$. Thus $b_k + c_{r_i,j}n > b_{q'} + c_{r_i-h',j}n + n$. This is impossible since B_{ij} is an r-antichain of w. So we have $h > h'$. Thus we have $b_q = b_{r_i-k+q,j}$ if $1 \le q \le k$. Similarly we see $b_q = b_{q-k,j}$ if $k < q \le r_i$.

(b) The assertion follows from the fact that B_{ij} is an r-antichain and (a).

The lemma is proved.

Now we are going to show that each element in $\Gamma_\lambda \cap \Gamma_\lambda^{-1}$ has a complete r-antichain family. The way is long.

Definition 5.1.5. We say that $w \in W$ is positive if $w(m)$ is positive for any positive integer m.

Lemma 5.1.6. *Let $w \in \Gamma_\lambda \cap \Gamma_\lambda^{-1}$ be positive. If $w(1) + \cdots + w(n) = 1 + \cdots + n$, then $w = w_\lambda$.*

Proof. By Lemma 2.5.1 we only need to show that $w(\Lambda_j) = \Lambda_j$ for all $j = 1, ..., r$. If the result is not true, we can find $1 \leq j \leq r$ such that $w(\Lambda_i) = \Lambda_i$ for $i = 1, 2, ..., j-1$ and $w(\Lambda_j) \neq \Lambda_j$. Then there exists a in Λ_j such that $w(a) = b$ is in Λ_k for some $r \geq k > j$.

Note that w^{-1} is also positive since $1 \leq w(m) \leq n$ whenever $1 \leq m \leq n$. By Lemma 5.1.1 we have

$$w^{-1}(b) = a > w^{-1}(a_{k-1,\lambda_{k-1}}) > \cdots > w^{-1}(a_{1,\lambda_1}) > 0.$$

Since a is in Λ_j, by our assumption on w, all the k elements in the above sequences are contained in the union of $\Lambda_1, ..., \Lambda_j$. Now $k > j$, we can find some $1 \leq i \leq j$ such that Λ_i contains at least two elements in the above sequence. This contradicts Corollary 2.4.2 since the above sequence is an r-antichain of w^{-1} and Λ_i is a d-chain of w.

Therefore $w(\Lambda_j) = \Lambda_j$ for all $j = 1, 2, ..., r$. The lemma is proved.

Lemma 5.1.7. *Let $w \in \Gamma_\lambda \cap \Gamma_\lambda^{-1}$ be positive. Let $a_{ij} \in \Lambda_i$ be such that $w(a_{ij}) > n$ and $w(a) \leq n$ if $a_{ij} < a \leq n$. Then $w(a) = w_\lambda(a)$ if $a_{ij} < a \leq n$.*

Proof. Assume $a_{ij} < a \leq n$ and $a \in \Lambda_k$. We first show that $w(a)$ is in Λ_k. We use descend induction on k.

When $k = r$, by Lemma 5.1.1 and assumption on a we get

$$n \geq w(a) > w(a_{r-1,\lambda_{r-1}}) > \cdots > w(a_{1,\lambda_1}) > 0.$$

Using Corollary 2.4.2 we can see that $w(a)$ is in Λ_r and $w(a_{k,\lambda_k})$ is in Λ_k for $k = 1, 2, ..., r-1$.

Now suppose that $w(b)$ is in Λ_l if $b \in \Lambda_l$ for some $k < l \leq r$. Using Lemma 5.1.1 we get

$$n \geq w(a) > w(a_{k-1,\lambda_{k-1}}) > \cdots > w(a_{1,\lambda_1}) > 0.$$

Since $w(\Lambda_l) = \Lambda_l$ whenever $k < l \leq r$, using Corollary 2.4.2 we see that $w(a)$ is necessarily in Λ_k.

Now we show $w(a) = w_\lambda(a)$ if $a_{ij} < a \leq n$. If $a \in \Lambda_l$ for some $i < l \leq r$, we must have $w(a) = w_\lambda(a)$ since $w(\Lambda_l) = \Lambda_l$ and $w \in \Gamma_\lambda$.

Assume $a_{ij} < a \leq n$ and $a \in \Lambda_i$. We only need to show that the two sets

$$A = \{w(a) \mid a_{ij} < a < a_{i+1,1}\}$$

and

$$B = \{w_\lambda(a) \mid a_{ij} < a < a_{i+1,1}\}$$

are equal. If this is not true, then we can find $a_{ij} < a, b < a_{i+1,1}$ such that $w(a) \notin B$ and $w_\lambda(b) \notin A$. Since $w(\Lambda_l) = \Lambda_l$ if $l > i$, we have $w(a') = w_\lambda(b) + cn$ for some $1 \leq a' \leq a_{ij}$ and integer c. Since w is positive, c is non-negative. We have $w(a) > w_\lambda(b)$ since both $w(a)$ and $w_\lambda(b)$ are in Λ_i and each number in $\Lambda_{r_i} - B$ is greater than any number in B. But

$$w^{-1}(w(a)) = a > a_{ij} \geq a' - cn = w^{-1}(w_\lambda(b)).$$

This contradicts that $w^{-1} \in \Gamma_\lambda$.

The lemma is proved.

Let $w \in \Gamma_\lambda \cap \Gamma_\lambda^{-1}$ and $1 \leq a \leq n$. Assume that $w(a) > n$ and $w(b) \leq n$ for all $a < b \leq n$. By Lemma 5.1.1 we see that a is Λ_{r_i} for some r_i, see the beginning of this section for the definition of r_i.

Lemma 5.1.8. *Let $w \in \Gamma_\lambda$ be positive and $a_{r_ij} \in \Lambda_{r_i}$. We can find $\xi_k \in \Lambda_k$ for $k = 1, 2, ..., r_i - 1$ such that*

$$w(a_{r_ij}) > w(\xi_{r_{i-1}}) > \cdots > w(\xi_2) > w(\xi_1) > w(a_{r_ij}) - n.$$

Proof. Since $w \in \Gamma_\lambda$ and $\lambda_{r_i} > \lambda_{r_i+1}$, we can find a d-antichain family set A of w of index λ_{r_i} with cardinality $\mu_1 + \cdots + \mu_{\lambda_{r_i}}$. By Prop. 2.4.4 we may assume that A is included in $\{1, 2, ..., n\}$. Let $D_1, ..., D_{\lambda_{r_i}}$ be the d-antichains of w in A. Since Λ_k is a d-chain of w for any k, using Lemma 2.4.1 we see that A contains at most λ_{r_i} elements of Λ_k for any k. Since A has cardinality $\mu_1 + \cdots + \mu_{\lambda_{r_i}}$, A contains exactly λ_{r_i} elements of Λ_k if $1 \le k \le r_i$ and contains all elements in Λ_k if $r_i \le k \le r$. Thus each d-antichain D_h $(1 \le h \le \lambda_{r_i})$ contains exactly one element of Λ_k if $1 \le k \le r_i$ and we can find a d-antichain D_l $(1 \le l \le \lambda_{r_i})$ that contains a_{r_ij}. For $1 \le k < r_i - 1$, let ξ_k be the unique element in $D_l \cap \Lambda_k$. Then the chosen ξ_k's satisfy our requirement. The lemma is proved.

5.1.9. Let $w \in \Gamma_\lambda \cap \Gamma_\lambda^{-1}$ be positive and $a_{r_ij} \in \Lambda_{r_i}$. Assume that $w(a_{r_ij}) > n$ and $w(b) \le n$ for all $a_{r_ij} < b \le n$. We want to define a positive element u in $\Gamma_\lambda \cap \Gamma_\lambda^{-1}$ with $u(1) + \cdots + u(n) = w(1) + \cdots + w(n) - n$.

Set $w(a_{k,\lambda_k+1}) = -\infty$ for all k. Choose $1 \le j_1 \le \lambda_1$ such that $w(a_{1j_1}) > w(a_{r_ij}) - n$ but $w(a_{1,j_1+1}) < w(a_{r_ij}) - n$ if $j_1 + 1 \le \lambda_1$. Then choose $2 \le j_k \le \lambda_k$ for $k = 1, 2, ..., r_i - 1$ such that

$$w(a_{k,j_k}) > w(a_{k-1,j_{k-1}}) > w(a_{k,j_k+1})$$

for $k = 2, 3, ..., r_i - 1$. According to Lemma 5.1.8, a_{kj_k} $(1 \le k \le r_i - 1)$ exists. Finally let $j_{r_i} = j$. For simplicity we set

$$a_k = a_{kj_k} \quad \text{for } k = 1, ..., r_i.$$

Lemma 5.1.10. *Keep the notation above. Then $w(a_{r_i}) > w(a_{r_i-1})$.*

Proof. According to Lemma 5.1.8 we can find ξ_k $(1 \le k \le r_i - 1)$ in Λ_k such that $w(a_{r_i}) > w(\xi_{r_i-1}) > \cdots > w(\xi_1) > w(a_{r_i}) - n$. By the definition of a_k we have $w(\xi_k) \ge w(a_k)$ for $k = 1, 2, ..., r_i - 1$. Therefore $w(a_{r_i}) > w(a_{r_i-1})$. The lemma is proved.

Keep the notation in 5.1.9. Now we define u by

$$u(a) = \begin{cases} w(a_{r_i}) - n & \text{if } a = a_1, \\ w(a_{k-1}) & \text{if } a = a_k,\ 2 \le k \le r_i, \\ w(a) & \text{if } a \not\equiv a_k (\text{mod } n), \quad \text{for all } 1 \le k \le r_i. \end{cases}$$

Clearly u is positive and $u(1) + \cdots + u(n) = w(1) + \cdots + w(n) - n$.

Lemma 5.1.11. *Let u be as above. Then u is in $\Gamma_\lambda \cap \Gamma_\lambda^{-1}$.*

Proof. We shall prove (1) $l(u) = l(uw_\lambda) + l(w_\lambda)$, (2) $l(u^{-1}) = l(u^{-1}w_\lambda) + l(w_\lambda)$, (3) $\lambda(u) = \lambda$.

Write

$$w(a_k) = b_k + c_k n, \qquad 1 \le b_k \le n, \quad c_k \in \mathbb{Z}.$$

Then $b_k \in \Lambda_{q_k}$ for some $1 \leq q_k \leq r_i$, since by Lemma 5.1.7, $w(\Lambda_q) = \Lambda_q$ if $r_i < q \leq r$. We also have $c_{r_i} \geq 1$ since $w(a_{r_i}) > n$.

By Lemma 5.1.10 and the definition of $w(a_k)$, we see that $w(a_1), ..., w(a_{r_i})$ form an r-antichain of w. Note that $w \in \Gamma_\lambda \cap \Gamma_\lambda^{-1}$. By Lemma 5.1.7, $w(\Lambda_q) = w_\lambda(\Lambda_q) = \Lambda_q$ whenever $q > r_i$. Using Corollary 2.4.2 we see that each Λ_q $(1 \leq q \leq r_i)$ contains exactly one b_k $(1 \leq k \leq r_i)$. As in the proof of Lemma 5.1.4 we get

$(*)$ If b_{r_i} is in Λ_k, then (a) b_{r_i-h} is in Λ_{k-h} for all $1 \leq h < k$, and b_{r_i-k-h} is in Λ_{r_i-h} if $0 \leq h < r_i - k$; (b) $c_{r_i} = \cdots = c_{r_i-k+1}$, and $c_{r_i-k} = \cdots = c_1 = c_{r_i} - 1$.

(1) By the definition of u and using Lemma 2.5.1, to check $l(u) = l(uw_\lambda) + l(w_\lambda)$, we only need to verify that $w(a_{r_i-1}) > w(a_{r_i,j+1})$. If b_{r_i} is not in Λ_1, then $c_{r_i-1} = c_{r_i} \geq 1$. In this case we have

$$w(a_{r_i-1}) > w(a_{r_i,j+1}) = w_\lambda(a_{r_i,j+1}).$$

If b_{r_i} is in Λ_1, then b_{r_i-1} is in Λ_{r_i} and $c_{r_i-1} = c_{r_i} - 1$. Since

$$w^{-1}(b_{r_i-1}) = a_{r_i-1} - (c_{r_i} - 1)n < w^{-1}(w(a_{r_i,j+1})) = a_{r_i,j+1},$$

$w^{-1} \in \Gamma_\lambda$ and $w(a_{r_i,j+1}) = w_\lambda(a_{r_i,j+1})$ is in Λ_{r_i} (cf. Lemma 5.1.7), we have $b_{r_i-1} > w(a_{r_i,j+1})$. Therefore

$$w(a_{r_i-1}) = b_{r_i-1} + (c_{r_i} - 1)n > w(a_{r_i,j+1}),$$

We have proved that $l(u) = l(uw_\lambda) + l(w_\lambda)$.

(2) Now we show that $l(u^{-1}w_\lambda) = l(u^{-1}) - l(w_\lambda)$. We have

$$u^{-1}(b) = \begin{cases} a_1 - (c_{r_i} - 1)n & \text{if } b = b_{r_i}, \\ a_{k+1} - c_k n & \text{if } b = b_k,\ 1 \leq k \leq r_i - 1, \\ w^{-1}(b) & \text{if } b \not\equiv b_k (\text{mod } n), \quad \text{for all } 1 \leq k \leq r_i. \end{cases}$$

Let $b > b'$ be in the same Λ_q for some q. We need to show that $u^{-1}(b) < u^{-1}(b')$. Note that $w^{-1}(b) < w^{-1}(b')$ since $w^{-1} \in \Gamma_\lambda$. When both b, b' are different from any b_k, we have

$$u^{-1}(b) = w^{-1}(b) < w^{-1}(b') = u^{-1}(b').$$

If $b > b' = b_k$ for some k, then $b \neq b_h$ for any $1 \leq h \leq r_i$ since b, b' are in the same Λ_q. Then we have

$$u^{-1}(b') = u^{-1}(b_k) > w^{-1}(b_k) > w^{-1}(b) = u^{-1}(b).$$

Now suppose that $b_k = b > b'$. Then $b' \neq b_h$ for any $1 \leq h \leq r_i$ since b, b' are in the same Λ_q. Since $w^{-1} \in \Gamma_\lambda$ we have

$$w^{-1}(b_k) = a_k - c_k n < w^{-1}(b') = a - cn,$$

where $1 \leq a \leq n$ and $c \in \mathbb{Z}$. Thus $c_k \geq c$.

Assume that $c_k > c$. When $1 \leq k \leq r_i - 1$, we have

$$u^{-1}(b_k) = a_{k+1} - c_k n < a - cn = w^{-1}(b') = u^{-1}(b')$$

.

Now suppose that $k = r_i$. We have

$$u^{-1}(b_k) = u^{-1}(b_{r_i}) = a_1 - (c_{r_i} - 1)n.$$

We claim that $a_1 < a$. Assume that $a \in \Lambda_l$. If $l > 1$ we of course have $a_1 < a$. Now suppose that $l = 1$. To see that $a_1 < a$ in this case it suffices to show that

$w(a) < w(a_1)$ since a, a_1 are in Λ_1 and $w \in \Gamma_\lambda$. If $b_{r_i} = b_k$ is in Λ_{r_i}, by $(*)$ we have $c_1 = c_{r_i}$. In this case we have $w(a) = b' + cn < w(a_1) = b_1 + c_1 n$ since $c < c_k = c_{r_i} = c_1$. If $b_{r_i} \in \Lambda_h$ for some $1 \leq h \leq r_i - 1$, then $b_1 \in \Lambda_{h+1}$, so $b_1 > b_{r_i} > b'$. In this case we also have $w(a) = b' + cn < w(a_1) = b_1 + c_1 n$ since $c \leq c_{r_i} - 1 = c_k - 1 = c_1$. We have seen that $a_1 < a$ when $k = r_i$. Noting that $c_k - 1 \geq c$, therefore if $k = r_i$ we also have

$$u^{-1}(b_k) = a_1 - (c_k - 1)n < a - cn = u^{-1}(b').$$

Assume that $c_k = c$. Then $a_k < a$. Suppose $a \in \Lambda_l$. Since $b = b_k, b'$ are in the same Λ_q, by $(*)$, we have

$$w(a_{k-1}) = b_{k-1} + c_{k-1}n < w(a) = b' + cn < w(a_k) = b_k + c_k n.$$

By the definition of a_k, we see that a is not in Λ_k. Thus $1 \leq k < l$. When $l > k+1$, we have $a > a_{k+1}$, so in this case

$$u^{-1}(b_k) = a_{k+1} - c_k n < a - cn = w^{-1}(b') = u^{-1}(b').$$

If $l = k + 1$, since

$$w(a_{k+1}) = b_{k+1} + c_{k+1}n > w(a_k) = b_k + c_k n > b' + cn = w(a)$$

and $w \in \Gamma_\lambda$, we have $a > a_{k+1}$. In this case we also have

$$u^{-1}(b_k) = a_{k+1} - c_k n < a - cn = w^{-1}(b') = u^{-1}(b').$$

We have showed that $l(u^{-1}w_\lambda) = l(u^{-1}) - l(w_\lambda)$.

(3) Now we prove $\lambda(u) = \lambda$. By (1) we see $\lambda(u) \geq \lambda$. If we can show $\lambda(u) \leq \lambda$, then we are forced to have $\lambda(u) = \lambda$.

Let $\{A_1, A_2, ..., A_j\}$ be an r-chain family of u (see §2.2 for definition). If we can construct an r-chain family $\{B_1, B_2, ..., B_j\}$ of w such that the cardinalities of $A_1 \cup \cdots \cup A_j$ and $B_1 \cup \cdots \cup B_j$ are the same, then we are done since it implies $\lambda(u) \leq \lambda(w) = \lambda$. Note that $u(a_1), ..., u(a_m)$ form an r-antichain of u. According to Corollary 2.4.2, we have

($\star$1) each A_k contains at most one element of the set $H = \{w(a_m) + ln \mid 1 \leq m \leq r_i,\ l \in \mathbb{Z}\} = \{u(a_m) + ln \mid 1 \leq m \leq r_i,\ l \in \mathbb{Z}\}$.

Since $A_1, ..., A_j$ form an r-chain family of u, we have

($\star$2) for any $1 \leq m \leq r_i$, at most one element of the set $H_m = \{w(a_m) + ln \mid l \in \mathbb{Z}\}$ is contained in the union $A_1 \cup \cdots \cup A_j$.

Assume that A_k contains some element of H if $1 \leq k \leq h$ and A_k does not contain any element of H if $h+1 \leq k \leq j$. Then $A_{h+1}, ..., A_j$ are also r-chains of w. Set $B_k = A_k$ if $h + 1 \leq k \leq j$.

Now we consider the r-chains $A_1, ..., A_h$ of u. Assume that $w(a_{m_k}) + l_k n$ ($1 \leq k \leq h$) is contained in A_k. It is no harm to assume $l_k = 0$ for $k = 1, ..., h$. Otherwise we may replace A_k by $\{u(a) - l_k n \mid u(a) \in A_k\}$. We may further assume $m_h > m_{h-1} > \cdots > m_1$.

Let

$$u(b_{k1}) > \cdots > u(b_{kq_k}) > u(a_{m_k}) > u(b_{k,q_k+2}) > \cdots$$

be the elements in A_k. We have two cases.

Case 1. For any $1 \leq k \leq h$, if b_{kq_k} exists, (that is, $u(a_{m_k})$ is not the greatest number in A_k,) then b_{kq_k} is in Λ_{m_k}.

Case 2. We can find some $1 \le k \le h$ such that b_{kq_k} exists and b_{kq_k} is not in Λ_{m_k}.

Set $a_{r_i+1} = a_1 + n$.

Assume that we are in case 1. By ($\star$1), ($\star$2), the definition of a_{m_k} and Lemma 5.1.10, we see $u(b_{kq_k}) > u(a_{m_k+1}) > u(a_{m_k})$ if $u(b_{kq_k})$ exists. Let B_k $(1 \le k \le h)$ be the set consisting of the elements

$$u(b_{k1}) > \cdots > u(b_{kq_k}) > u(a_{m_k+1}) > u(b_{k,q_k+2}) > \cdots .$$

Note that Λ_{m_k} is a d-chain of u. By the definition of u we see that $B_1, ..., B_h$ are r-chains of w. Clearly $B_1, ..., B_j$ form an r-chain family of w and the cardinalities of $A_1 \cup \cdots \cup A_j$ and $B_1 \cup \cdots \cup B_j$ are the same.

Now assume that we are in case 2. Let k $(1 \le k \le h)$ be the maximal such that either b_{kq_k} does not exist or b_{kq_k} exists and is not in Λ_{m_k}. We claim that A_k is an r-chain of w. When $m_k > 1$, this is clear since Λ_{m_k-1} is a d-chain of w. When $m_k = 1$, we necessarily have $k = 1$ since $m_1 < \cdots < m_{h-1} < m_h$. Noting that $u(a_1)$ is positive and $u(b) = w(b) = w_\lambda(b)$ if $a_{r_ij} < b \le n$ (cf. Lemma 5.1.7), we see $b_{1q_1} < a_{r_ij} - n$ since b_{1q_1} is not in Λ_1 when b_{1q_1} exists. Therefore A_k is also an r-chain of w when $k = 1$.

If $k > 1$, we would like to construct two r-chains A'_{k-1}, A'_k of w from A_{k-1}, A_k such that $A'_{k-1} \cup A'_k$ and $A_{k-1} \cup A_k$ have the same cardinality.

If $b_{k-1,q_{k-1}}$ does not exist or $b_{k-1,q_{k-1}}$ is not in $\Lambda_{m_{k-1}}$ when it exists, we set $A'_{k-1} = A_{k-1}$ and $A'_k = A_k$.

Now assume that $b_{k-1,q_{k-1}}$ exists and is in $\Lambda_{m_{k-1}}$.

If $m_{k-1} < m_k - 1$, we set $A'_{k-1} = (A_{k-1} - \{u(a_{m_{k-1}})\}) \cup \{u(a_{m_{k-1}+1})\}$ and $A'_k = A_k$.

Suppose that $m_{k-1} = m_k - 1$ and $b_{k-1,q_{k-1}}$ is in $\Lambda_{m_{k-1}}$. If b_{kq_k} exists and is in $\Lambda_{m_{k-1}}$, we choose $1 \le p_k \le q_k$ and $1 \le p_{k-1} \le q_{k-1}$ such that both $b_{k-1,p_{k-1}}$ and b_{kp_k} are in $\Lambda_{m_{k-1}} = \Lambda_{m_k-1}$ but neither $b_{k-1,p_{k-1}-1}$ nor b_{k,p_k-1} is in $\Lambda_{m_{k-1}}$. We can move all elements in the intersection of $u(\Lambda_{m_k-1}) = u(\Lambda_{m_{k-1}})$ and $A_{k-1} \cup A_k$ to A_k and form two sets A'_k, A'_{k-1} as follows. If $u(b_{kp_k}) > u(b_{k-1,p_{k-1}})$, we set

$$A'_k = A_k \cup \{u(b_{k-1,p_{k-1}}), ..., u(b_{k-1,q_{k-1}})\},$$
$$A'_{k-1} = A_{k-1} - \{u(b_{k-1,p_{k-1}}), ..., u(b_{k-1,q_{k-1}})\}.$$

If $u(b_{kp_k}) < u(b_{k-1,p_{k-1}})$, let A'_k be the set consisting of

$$u(b_{k-1,1}),\ u(b_{k-1,2}),\ ...,\ u(b_{k-1,q_{k-1}}),$$
$$u(b_{kp_k}),\ ...,\ u(b_{kq_k}),\ u(a_{m_k}),\ u(b_{k,q_k+2}),\ ...,$$

and let A'_{k-1} be the set consisting of

$$u(b_{k1}),\ ...,\ u(b_{k,p_k-1}),\ u(a_{m_{k-1}}),\ u(b_{k-1,q_{k-1}+2}),\ ....$$

Suppose that $m_{k-1} = m_k - 1$ and $b_{k-1,q_{k-1}}$ is in $\Lambda_{m_{k-1}}$. If b_{kq_k} does not exist or b_{kq_k} exists but b_{kq_k} is not in $\Lambda_{m_{k-1}}$, we choose $1 \le p_{k-1} \le q_{k-1}$ such that $b_{k-1,p_{k-1}}$ is in $\Lambda_{m_{k-1}} = \Lambda_{m_k-1}$ but $b_{k-1,p_{k-1}-1}$ is not in $\Lambda_{m_{k-1}}$. We can move all elements in the intersection of $u(\Lambda_{m_k-1}) = u(\Lambda_{m_{k-1}})$ and A_{k-1} to A_k and form two sets A'_k, A'_{k-1} as follows. If b_{kq_k} does not exist or b_{kq_k} exists and $u(b_{kq_k}) > u(b_{k-1,p_{k-1}})$, we set

$$A'_k = A_k \cup \{u(b_{k-1,p_{k-1}}), ..., u(b_{k-1,q_{k-1}})\},$$
$$A'_{k-1} = A_{k-1} - \{u(b_{k-1,p_{k-1}}), ..., u(b_{k-1,q_{k-1}})\}.$$

If b_{kq_k} exists and $u(b_{kq_k}) < u(b_{k-1,p_{k-1}})$, let A'_k be the set consisting of

$$u(b_{k-1,1}),\ u(b_{k-1,2}),\ ...,\ u(b_{k-1,q_{k-1}}),\ u(a_{m_k}),\ u(b_{k,q_k+2}),\ ...,$$

and let A'_{k-1} be the set consisting of

$$u(b_{k1}),\ ...,\ u(b_{kq_k}),\ u(a_{m_{k-1}}),\ u(b_{k-1,q_{k-1}+2}),\ ....$$

It is easy to see that in any case A'_{k-1} and A'_k are r-chains of w. Moreover $A'_{k-1} \cup A'_k$ and $A_{k-1} \cup A_k$ have the same cardinality. If $k-1 > 1$, for the pair A'_{k-1}, A_{k-2}, we apply the same process, then we get two r-chains A''_{k-1}, A'_{k-2} of w such that $A''_{k-1} \cup A'_{k-2}$ and $A'_{k-1} \cup A_{k-2}$ have the same cardinality. Continuing this procedure, we get k r-chains $A'_k, A''_{k-1}, ..., A''_2, A'_1$ of w. From the construction we see that each of the r-chains $A'_k, A''_{k-1}, ..., A''_2, A'_1$ of w contains some $u(a_m)$ with $1 \leq m \leq m_k$.

When $h = k > 1$, we set $B_k = A'_k, B_{k-1} = A''_{k-1}, ..., B_2 = A''_2, B_1 = A'_1$.

When $h = k = 1$, we set $B_1 = A_1$.

Now assume $1 \leq k < h$. Recall that we are in case 2 and A_k is also an r-chain of w. When $k > 1$, we have constructed k r-chains $A'_k, A''_{k-1}, ..., A''_2, A'_1$ of w. If $k = 1$ we set $A'_1 = A_1$. By our assumption on k, for any $k+1 \leq l \leq h$, b_{lq_l} exists and b_{lq_l} is in Λ_l. If $m_h < r_i$ or $u(a_1)$ is not in A'_1, then let B_l $(k+1 \leq l \leq h)$ be the set consisting of the elements

$$u(b_{l1}) > \cdots > u(b_{lq_l}) > u(a_{m_l+1}) > u(b_{l,q_l+2}) > \cdots .$$

And we set $B_k = A'_k, B_{k-1} = A''_{k-1}, ..., B_2 = A''_2, B_1 = A'_1$.

Assume that $m_h = r_i$ and $u(a_1)$ is in A'_1. We construct two r-chains A''_1, A'_h of w as follows.

Note that b_{hq_h} exists and is in $\Lambda_{m_h} = \Lambda_{r_i}$. Let

$$u(c_1) > \cdots > u(c_q) > u(a_1) > u(c_{q+2}) > \cdots$$

be the elements in A'_1, and

$$u(d_1) > \cdots > u(d_{q'}) > u(a_{r_i}) > u(d_{q'+2}) > \cdots$$

be the elements in A_h. We have $d_{q'} = b_{hq_h}$.

Note that $u(a_1)$ is positive. We have $c_q < a_{r_ij} - n$ if $u(c_q)$ exists, since A'_1 is an r-chain of w and $u(b) = w(b) = w_\lambda(b)$ if $a_{r_ij} < b \leq n$ (cf. Lemma 5.1.7). If c_q exists and $c_q + n$ is in Λ_{r_i}, we choose $1 \leq p \leq q$ and $1 \leq p' \leq q'$ such that both $c_p + n$ and $d_{p'}$ are in $\Lambda_{r_i} = \Lambda_{m_h}$ but neither $c_{p-1} + n$ nor $d_{p'-1}$ is in Λ_{r_i}. If $u(c_p + n) > u(d_{p'})$, we set

$$A''_1 = A'_1 \cup \{u(d_{p'} - n),\ ...,\ u(d_{q'} - n)\},$$
$$A'_h = A_h - \{u(d_{p'}),\ ...,\ u(d_{q'})\}.$$

If $u(c_p + n) < u(d_{p'})$, let A''_1 be the set consisting of

$$u(d_1 - n),\ u(d_2 - n),\ ...,\ u(d_{q'} - n),\ u(c_p),\ ...,\ u(c_q),\ u(a_1),\ u(c_{q+2}),\ ...,$$

and let A'_h be the set consisting of

$$u(c_1 + n),\ ...,\ u(c_{p-1} + n),\ u(a_{m_h}),\ u(d_{q'+2}),\ ....$$

If c_q does not exist or c_q exists but $c_q + n$ is not in Λ_{r_i}, we choose $1 \leq p' \leq q'$ such that $d_{p'}$ is in $\Lambda_{r_i} = \Lambda_{m_h}$ but $d_{p'-1}$ is not in Λ_{r_i}. If c_q does not exists or c_q exists and $u(c_q + n) > u(d_{p'})$, we set

$$A''_1 = A'_1 \cup \{u(d_{p'} - n),\ ...,\ u(d_{q'} - n)\},$$

$$A'_h = A_h - \{u(d_{p'}),\ ...,\ u(d_{q'})\}.$$

If c_q exists and $u(c_q+n) < u(d_{p'})$, let A''_1 be the set consisting of

$$u(d_1-n),\ u(d_2-n),\ ...,\ u(d_{q'}-n),\ u(a_1),\ u(c_{q+2}),\ ...,$$

and let A'_h be the set consisting of

$$u(c_1+n),\ ...,\ u(c_q+n),\ u(a_{m_h}),\ u(d_{q'+2}),\ ....$$

It is easy to see that in any case A''_1 and A'_h are r-chains of w. Moreover $A''_1 \cup A'_h$ and $A'_1 \cup A_h$ have the same cardinality. If $k < h-1$, for the pair A'_h, A_{h-1}, we apply the same process as that for A_k, A_{k-1} (in the case $k > 1$), then we get two r-chains A''_h, A'_{h-1} of w such that $A''_h \cup A'_{h-1}$ and $A'_h \cup A_{h-1}$ have the same cardinality. Continuing this procedure, we get $h-k$ r-chains $A''_h, A''_{h-1}, ..., A''_{k+2}, A'_{k+1}$ of w. From the construction we see that each of the $h-k$ r-chains of w contains some $u(a_m)$ with $m_k < m \le r_i$. We set $B_h = A''_h, ..., B_{k+2} = A''_{k+2}, B_{k+1} = A'_{k+1}$, and $B_k = A'_k, B_{k-1} = A''_{k-1}, ..., B_2 = A''_2, B_1 = A''_1$.

From the construction it is clear that in any case $B_1, ..., B_j$ form an r-chain family of w and the cardinalities of $A_1 \cup \cdots \cup A_j$ and $B_1 \cup \cdots \cup B_j$ are the same.

The lemma is proved.

Now we are in the position to state the main result of this section.

Theorem 5.1.12. *Each element of $\Gamma_\lambda \cap \Gamma_\lambda^{-1}$ has a complete r-antichain family.*

Proof. Let w be an element in $\Gamma_\lambda \cap \Gamma_\lambda^{-1}$. We need show that w has a complete r-antichain family. We show the result first in the case when w is positive (see 5.1.5 for definition), and then in general.

Suppose that w is positive. We use induction on the sum $E(w) = w(1) + \cdots + w(n)$. When $E(w) = 1 + \cdots + n$, by Lemma 5.1.6, $w = w_\lambda$. According to 2.4.6, the result is true

Now suppose that $E(w) > 1 + \cdots + n$ and the result is true if $u \in \Gamma_\lambda \cap \Gamma_\lambda^{-1}$ is positive and $E(u) < E(w)$. We can find r_i and j such that $w(a_{r_i j}) > n$ and $w(a) \le n$ whenever $a_{r_i j} < a \le n$. Let u be as in Lemma 5.1.11. Then $u \in \Gamma_\lambda \cap \Gamma_\lambda^{-1}$ is positive and $E(u) = E(w) - n < E(w)$. By induction hypothesis, u has a complete r-antichain family.

Keep the notation in 5.1.9. Let Z be a complete r-antichain family of u. We may require that the r-antichain in Z containing $u(a_{r_i})$ has length r_i for the reason explained below. If $u(a_{r_i}) > n$, then the r-antichain in Z containing $u(a_{r_i})$ has length r_i, since $u(\Lambda_q) = w(\Lambda_q) = \Lambda_q$ when $r_i < q \le r$. If $u(a_{r_i}) \le n$, by Lemma 5.1.7, we have $u(a) = w_\lambda(a)$ if $a_{r_i} \le a \le n$. Since $w(a_{r_i}) > n$, by Lemma 5.1.1, we see $\lambda_{r_i} - j \ge \lambda_{r_i+1}$. Thus at least one r-antichain A in Z that contains $u(a)$ for some $a_{r_i} \le a \le a_{r_i, \lambda_{r_i}}$ and has length r_i. Let B be the r-antichain in Z containing $u(a_{r_i})$. According to Lemma 5.1.4, $(A - \{u(a)\}) \cup \{u(a_{r_i})\}$ and $(B - \{u(a_{r_i})\}) \cup \{u(a)\}$ are r-antichains of u. Thus it is harmless to assume that the r-antichain B in Z containing $u(a_{r_i})$ has length r_i.

Assume that $\xi_k \in \Lambda_k$ $(1 \le k < r_i - 1)$ and $u(\xi_k)$ is in B for all k. Note that $u(a_{r_1}), ..., u(a_1)$ form an r-antichain of u and $u(\Lambda_q) = \Lambda_q$ if $r_i < q \le r$. Let B_k be the r-antichain in Z containing $u(a_k)$. According to Lemma 5.1.4 and the assertion $(*)$ in the proof of Lemma 5.1.1, the following sets are also r-antichains: $B' = \{u(a_{r_i}), ..., u(a_1)\}$, $B'_k = (B_k - \{u(a_k)\}) \cup \{u(\xi_k)\}$ $(1 \le k \le r_i - 1)$. Replacing

B, B_k $(1 \le k \le r_i - 1)$ in Z by B', B'_k $(1 \le k \le r_i - 1)$, respectively, we get a complete r-antichain family Y of u such that the r-antichain $B' = \{u(a_{r_i}), ..., u(a_1)\}$ of u is in Y. Replacing B' in Y by the r-antichain $\{w(a_{r_i}), ..., w(a_1)\}$ of w, we then get a complete r-antichain family of w.

We have showed the result for positive elements in $\Gamma_\lambda \cap \Gamma_\lambda^{-1}$. For any element w in $\Gamma_\lambda \cap \Gamma_\lambda^{-1}$, we can find a positive integer a such that $\omega^{an} w \in \Gamma_\lambda \cap \Gamma_\lambda^{-1}$ is positive. Noting that any d-antichain of $\omega^{an} w$ is also a d-antichain of w, thus w has a complete r-antichain family since $\omega^{an} w$ has one.

The theorem is proved.

In next section we use this result to establish a bijection between $\Gamma_\lambda \cap \Gamma_\lambda^{-1}$ and $\mathrm{Dom}(F_\lambda)$.

5.2. A map from $\Gamma_\lambda \cap \Gamma_\lambda^{-1}$ to Dom(F_λ)

In this section we establish the bijection between $\Gamma_\lambda \cap \Gamma_\lambda^{-1}$ and $\mathrm{Dom}(F_\lambda)$ by means of complete r-antichain family.

5.2.1. Let w be in $\Gamma_\lambda \cap \Gamma_\lambda^{-1}$. By Theorem 5.1.12, w has a complete r-antichain family Z. In 5.1.2 we have seen that Z contains $n_i (= \lambda_{r_i} - \lambda_{r_{i+1}})$ r-antichains of w of length r_i. Let $B_{i1}, ..., B_{in_i}$ be the r-antichains in Z of length r_i. Let

$$b_{r_i,j} + c_{r_i,j} n > b_{r_i-1,j} + c_{r_i-1,j} n > \cdots > b_{1,j} + c_{1,j} n$$

be elements in B_{ij}, where $1 \le b_{k,j} \le n$ and $c_{k,j} \in \mathbb{Z}$ for all $1 \le k \le r_i$ and $1 \le j \le n_i$. It is no harm to assume that

$$b_{r_i,1} + c_{r_i,1} n > b_{r_i,2} + c_{r_i,2} n > \cdots > b_{r_i,n_i} + c_{r_i,n_i} n.$$

Define $\varepsilon_{ij}(Z) = \varepsilon(B_{ij}) = \sum_{1 \le k \le r_i} c_{k,j}$. From Lemma 5.1.4 and our assumption on the arrangement $B_{i1}, B_{i2}, ..., B_{in_i}$, we see $\varepsilon_{ij}(Z) = \varepsilon(B_{ij}) \ge \varepsilon_{i,j+1}(Z) = \varepsilon(B_{i,j+1})$ if $j + 1 \le n_i$. Thus we have defined an element

$$\varepsilon(Z) = (\varepsilon_{11}(Z), ..., \varepsilon_{1n_1}(Z), ..., \varepsilon_{p1}(Z), ..., \varepsilon_{pn_p}(Z)) \in \mathrm{Dom}(F_\lambda).$$

We will show that $\varepsilon(Z)$ is independent of the choice of the complete r-antichain family Z. To do this, we construct a particular complete r-antichain family of w in next subsection.

5.2.2. Assume that $A = (x_{11}, x_{12}, ..., x_{1\lambda_1}, x_{21}, x_{22}, ..., x_{2\lambda_2}, ..., x_{r1}, ..., x_{r\lambda_r})$ is in $\mathbb{Z}^n$. We call that A is **λ-admissible** if

(1) any two components of A are different,

(2) $x_{i1} > x_{i2} > \cdots > x_{i\lambda_i}$ for $i = 1, ..., r$,

(3) given $1 \le h < i \le r$, $1 \le j \le \lambda_i$, we have $x_{ij} > x_{h,\lambda_h - \lambda_i + j}$, in particular, we have $x_{ij} > x_{i-1,\lambda_{i-1} - \lambda_i + j}$.

Let $A = (x_{11}, ..., x_{1\lambda_1}, ..., x_{r1}, ..., x_{r\lambda_r})$ be a λ-admissible element of $\mathbb{Z}^n$. We define $\varepsilon_{k,i,j}$ inductively for all $1 \le k \le r_i$, $1 \le i \le p$, and $0 \le j \le n_i$, (see the beginning of section 5.1 for the definition of r_i, p, n_i). First we define $\varepsilon_{k,i,0} = \infty$ for all k, i. Let x be the unique greatest number among all components of A. Then $x = x_{r_i,j}$ for some $1 \le i \le p$ and $1 \le j \le \lambda_{r_i}$. Let $\varepsilon_{r_i,i,1} = x_{r_i,j}$. Assume that we have defined $\varepsilon_{k,i,1}$ for some $k \le r_i$. We then define $\varepsilon_{k-1,i,1}$ to be the maximal number in $\{x_{k-1,q} \mid x_{k-1,q} < \varepsilon_{k,i,1},\ 1 \le q \le \lambda_{k-1}\}$.

Suppose that we have defined $\varepsilon_{l,h,q}$ for all h in $\{1, 2, ..., p\}$ and $1 \leq l \leq r_h$, $0 \leq q \leq b_h$. Here b_h are some nonnegative integers, (note that we have defined $\varepsilon_{l,h,0} = \infty$ for all l, h). Let $B \in \mathbb{Z}^{n'}$ be obtained from A by removing all the components $\varepsilon_{l,h,q}$ $(1 \leq q \leq b_h)$ for some n'. Let $g_{p'} = x_{p'q'}$ be the greatest number among all components of B. Then it is easy to see $p' = r_{h'}$ for some h'. Suppose that we have defined g_k for $k \leq p' = r_{h'}$. Then we define g_{k-1} to be the maximal number in

$$\{x_{k-1,q'} \text{ is a component of } B \mid x_{k-1,q'} < g_k,\ 1 \leq q' \leq \lambda_{k-1}\}.$$

Then we define $\varepsilon_{k,h',b_{h'}+1} = g_k$ for any $1 \leq k \leq r_{h'}$. Continuing this way we define all $\varepsilon_{k,i,j}$ for $1 \leq k \leq r_i,\ 1 \leq i \leq p$ and $1 \leq j \leq n_i$.

Example: Let $\lambda = (4, 3, 2, 2)$. Then $r_1 = 1,\ r_2 = 2,\ r_3 = 4$. And $A = (11, 7, 4, 3; 12, 6, 5; 10, 8; 14, 9)$ is λ-admissible. We have

$$\varepsilon_{4,3,1} = 14,\quad \varepsilon_{3,3,1} = 10,\quad \varepsilon_{2,3,1} = 6,\quad \varepsilon_{1,3,1} = 4;$$

$$\varepsilon_{4,3,2} = 9,\quad \varepsilon_{3,3,2} = 8,\quad \varepsilon_{2,3,2} = 5,\quad \varepsilon_{1,3,2} = 3;$$

$$\varepsilon_{2,2,1} = 12,\quad \varepsilon_{1,2,1} = 11; \qquad \varepsilon_{1,1,1} = 7.$$

Let $w \in \Gamma_\lambda \cap \Gamma_\lambda^{-1}$. Set $x_{ij} = w(e_{i-1} + j)$ for any $1 \leq i \leq r$ and $1 \leq j \leq \lambda_i$. According to Lemma 5.1.1 we see that $A = (x_{11}, ..., x_{1\lambda_1}, ..., x_{r1}, ..., x_{r\lambda_r}) \in \mathbb{Z}^n$ is λ-admissible. Thus we can define $\varepsilon_{k,i,j}$. To indicate its relation with w, we shall denote it by $\varepsilon_{k,i,j}(w)$. We would like to understand the properties of $\varepsilon_{k,i,j}(w)$.

Lemma 5.2.3. *Let w be in $\Gamma_\lambda \cap \Gamma_\lambda^{-1}$. For given integers $1 \leq i \leq p$ and $1 \leq j \leq n_i$, the numbers $\varepsilon_{k,i,j}(w)$ $(1 \leq k \leq r_i)$ form an r-antichain of w of length r_i. Thus the r-antichains*

$$\{\varepsilon_{k,i,j}(w) \mid 1 \leq k \leq r_i\} \quad (1 \leq i \leq p,\ 1 \leq j \leq n_i)$$

of w form a complete r-antichain family, we denote it by Z_w.

Proof. For simplicity we write $\varepsilon_{k,i,j}$ instead of $\varepsilon_{k,i,j}(w)$ in this proof. Let Z be a complete r-antichain of w. If $\varepsilon_{r_i,i,j}$ is maximal among all $w(1), ..., w(n)$, using 5.1.2 (a), we see that the r-antichain B in Z containing $\varepsilon_{r_i,i,j}$ has length r_i. Moreover by the definition of $\varepsilon_{k,i,j}$ we see that the smallest number in B is not greater than $\varepsilon_{1,i,j}$. Thus $\varepsilon_{r_i,i,j} - n < \varepsilon_{1,i,j}$. So all $\varepsilon_{k,i,j}$ $(1 \leq k \leq r_i)$ form an r-antichain of w of length r_i.

In general, we consider the r-antichains of w in Z.

Let

$$I : w(\xi_s) > w(\xi_{s-1}) > \cdots > w(\xi_1)$$

$$J : w(\eta_t) > w(\eta_{t-1}) > \cdots > w(\eta_1)$$

be two r-antichains of w in Z. Then ξ_h and η_h are in Λ_h for all h. Suppose $w(\xi_s) > w(\eta_t)$. If

$$w(\xi_{u+1}) > w(\eta_u),\quad w(\eta_u) > w(\xi_u),\quad w(\eta_{u-1}) > w(\xi_{u-1}),$$
$$...,\quad w(\eta_{u-v}) > w(\xi_{u-v}),\quad w(\eta_{u-v-1}) < w(\xi_{u-v-1})$$

for some $0 \leq v < u \leq t, s-1$, then

$$\begin{aligned} I': \quad & w(\xi_s) > w(\xi_{s-1}) > \cdots > w(\xi_{u+1}) > w(\eta_u) \\ & \quad > w(\eta_{u-1}) > \cdots > w(\eta_{u-v}) > w(\xi_{u-v-1}) > \cdots > w(\xi_1), \end{aligned}$$

$$\begin{aligned} J': \quad & w(\eta_t) > w(\eta_{t-1}) > \cdots > w(\eta_{u+1}) > w(\xi_u) \\ & > w(\xi_{u-1}) > \cdots > w(\xi_{u-v}) > w(\eta_{u-v-1}) > \cdots > w(\eta_1) \end{aligned}$$

are two r-antichains of w. Replacing I, J by I', J' in Z respectively, we then get a new complete r-antichain family Z' of w. Continuing this process, finally we get a complete r-antichain family Z'' of w with the following property. If $I : w(\xi_s) > w(\xi_{s-1}) > \cdots > w(\xi_1)$ and $J : w(\eta_t) > w(\eta_{t-1}) > \cdots > w(\eta_1)$ are two r-antichains of w in Z'' and $w(\xi_s) > w(\eta_t)$, then $w(\xi_h) > w(\eta_h)$ for all $1 \leq h \leq s, t$ whenever $w(\xi_{h+1}) > w(\eta_h)$. By the definition of $\varepsilon_{k,i,j}$ we see that

$$\varepsilon_{r_i,i,j} > \varepsilon_{r_i-1,i,j} > \cdots > \varepsilon_{1,i,j}$$

is exactly an r-antichain in Z''. The lemma is proved.

Lemma 5.2.4. *Let $w \in \Gamma_\lambda \cap \Gamma_\lambda^{-1}$ and Z be a complete r-antichain family of w. Then the dominant weight $\varepsilon(Z)$ is independent of the choice of the complete r-antichain family Z of w and only depends on w. Thus we can denote the dominant weight by $\varepsilon(w)$ and denote $\varepsilon_{ij}(Z)$ by $\varepsilon_{ij}(w)$.*

Proof. Let I, J, I', J' be as in the proof of Lemma 5.2.3. We claim that $\varepsilon(I) = \varepsilon(I')$ and $\varepsilon(J) = \varepsilon(J')$. By Lemma 5.1.4, $\varepsilon(I)$ and $\varepsilon(I')$ are completely determined by $w(\xi_s)$. So we always have $\varepsilon(I) = \varepsilon(I')$. If $u \neq t$, then $w(\eta_t)$ is the maximal number in J and J' as well. By Lemma 5.1.4 we see in this case $\varepsilon(J) = \varepsilon(J')$. If $u = t$, then $s > t$. Write $w(\xi_t) = b_1 + c_1 n$ and $w(\eta_t) = b_2 + c_2 n$, where $1 \leq b_1, b_2 \leq n$ and $c_1, c_2 \in \mathbb{Z}$. Using Lemma 5.1.4 for the r-antichains I and I' we can find k such that both b_1, b_2 are in Λ_k and then $c_1 = c_2$. Using Lemma 5.1.4 we see $\varepsilon(J) = \varepsilon(J')$. Thus we have $\varepsilon(Z) = \varepsilon(Z') = \varepsilon(Z_w)$. The lemma is proved.

It seems that the number $\varepsilon_{ij}(w)$ has a strange property which now we are going to state. Define $\varepsilon'_{k,i,j}(w) = \varepsilon_{k,i,j}(w) - w_\lambda(f)$ if $\varepsilon_{k,i,j} = w(f)$.

Lemma 5.2.5. *$\varepsilon'_{k,i,j}(w) \geq \varepsilon'_{k,i,j+1}(w)$ if $j + 1 \leq n_i$.*

Proof. Let $\varepsilon_{k,i,j}(w) = w(e_{k-1} + f)$ and $\varepsilon_{k,i,j+1}(w) = w(e_{k-1} + g)$. Then

$$\varepsilon_{k,i,j}(w) - \varepsilon_{k,i,j+1}(w) \geq g - f = w_\lambda(e_{k-1} + f) - w_\lambda(e_{k-1} + g).$$

Therefore $\varepsilon'_{k,i,j}(w) \geq \varepsilon'_{k,i,j+1}(w)$. The lemma is proved.

Remark: It is likely that $\varepsilon_{ij}(w) = \frac{1}{n} \sum_{1 \leq k \leq r_i} \varepsilon'_{k,i,j}(w)$.

The following is the main result of this chapter.

Theorem 5.2.6. *Let w be in $\Gamma_\lambda \cap \Gamma_\lambda^{-1}$. We have*

(a) $$\varepsilon(w^{-1}) = (-\varepsilon_{1n_1}(w), ..., -\varepsilon_{11}(w), ..., -\varepsilon_{pn_p}(w), ..., -\varepsilon_{p1}(w)).$$

if $\varepsilon(w) = (\varepsilon_{11}(w), ..., \varepsilon_{1n_1}(w), ..., \varepsilon_{p1}(w), ..., \varepsilon_{pn_p}(w))$.

(b) *The map $\varepsilon : w \to \varepsilon(w)$ defines a bijection between $\Gamma_\lambda \cap \Gamma_\lambda^{-1}$ and* $\mathrm{Dom}(F_\lambda)$.

Proof. (a) Let $B = B_{ij} \in Z_w$ be an r-antichain of w of length r_i consisting of

$$b_{r_i} + c_{r_i} n,\ b_{r_i-1} + c_{r_i-1} n,\ \ldots,\ b_1 + c_1 n,$$

where all b_k are in $[1, n]$ and c_k are in $\mathbb{Z}$. Using Lemma 5.1.4 it is easy to see that all $w^{-1}(b_k) = w^{-1}(b_k + c_k n) - c_k n$ form an r-antichain B' of w^{-1} that is equivalent to $w^{-1}(B)$. (See Lemma 2.4.3 for the definition of equivalent r-antichains). All such r-antichains B' of course form a complete r-antichain family of w^{-1}. By definition we have

$$\varepsilon(B') = -c_1 - c_2 - \cdots - c_{r_i} = -\varepsilon(B).$$

By the definition of $\varepsilon(w)$ and of $\varepsilon(w^{-1})$ we see that (a) is true.

We will prove (b) in next section.

Before going further, we give two examples.

(1) Assume that $\lambda = (n)$. Then the dual μ of λ is (1,...,1) and $w_\lambda = w_0$ is the longest element of $W_0 =< s_1, ..., s_{n-1} >$. In this case we have

$$\Gamma_\lambda \cap \Gamma_\lambda^{-1} = \{w_0 x \mid x \in X^+\}$$

and $F_\lambda = GL_n(\mathbb{C})$. Therefore $\mathrm{Dom}(F_\lambda) = \mathbb{Z}^n_{\mathrm{dom}}$. Let w be in $\Gamma_\lambda \cap \Gamma_\lambda^{-1}$. Then w has a unique complete r-antichain family, which consists of $\{w(1)\}, ..., \{w(n)\}$. Assume that $w = w_0 x_1^{a_1} \cdots x_{n-1}^{a_{n-1}} x_n^{a_n}$, where $a_1, ..., a_{n-1} \in \mathbb{N}$ and $a_n \in \mathbb{Z}$. Then we have

$$\varepsilon(w) = (a_1 + a_2 + a_3 + \cdots + a_n,\ a_2 + a_3 + \cdots + a_n,\ \ldots,\ a_{n-1} + a_n,\ a_n),$$

which is in $\mathrm{Dom}(F_\lambda) = \mathbb{Z}^n_{\mathrm{dom}}$.

(2) Assume that $n \geq 3$ and $\lambda = (2, 1, ..., 1)$. Then the dual μ of λ is $(n-1$,1) and $w_\lambda = s_1$. In this case we have

$$\Gamma_\lambda \cap \Gamma_\lambda^{-1} = \{\omega^{an} s_1 (\omega s_1)^b,\ \ \omega^{an} s_1 (\omega^{-1} s_1)^b \mid a \in \mathbb{Z},\ b \in \mathbb{N}\}$$

and F_λ is isomorphic to $\mathbb{C}^* \times \mathbb{C}^*$. Therefore $\mathrm{Dom}(F_\lambda) = \mathbb{Z}^2$. We have

$$\varepsilon(\omega^{an} s_1 (\omega s_1)^b) = (a, a(n-1) + b) \in \mathbb{Z}^2 = \mathrm{Dom}(F_\lambda)$$

and

$$\varepsilon(\omega^{an} s_1 (\omega^{-1} s_1)^b) = (a, a(n-1) - b) \in \mathbb{Z}^2 = \mathrm{Dom}(F_\lambda).$$

5.3. Constructing elements of $\Gamma_\lambda \cap \Gamma_\lambda^{-1}$

In this section we will give a proof of Theorem 5.2.6 (b). To do this we need construct elements of $\Gamma_\lambda \cap \Gamma_\lambda^{-1}$. More precisely for a given dominant weight $\varepsilon = (\varepsilon_{11}, ..., \varepsilon_{1n_1}, ..., \varepsilon_{p1}, ..., \varepsilon_{pn_p})$ in $\mathrm{Dom}(F_\lambda)$ we will construct an element w_ε in $\Gamma_\lambda \cap \Gamma_\lambda^{-1}$ such that $\varepsilon(w_\varepsilon) = \varepsilon$. Then we show that the map ε is a bijection from $\Gamma_\lambda \cap \Gamma_\lambda^{-1}$ to $\mathrm{Dom}(F_\lambda)$.

First we assume that all ε_{ij} are nonnegative.

If $\varepsilon = (1, 0, ..., 0) \in \mathrm{Dom}(F_\lambda)$, then w_ε is defined by

$$w_\varepsilon(a) = \begin{cases} (i+1)\lambda_1 & \text{if } a = a_{i1},\ 1 \leq i \leq r_1 - 1, \\ \lambda_1 + n & \text{if } a = a_{r_1 1}, \\ w_\lambda(a) & \text{if } a \not\equiv a_{i1} (\mathrm{mod}\ n), \quad \text{for all } 1 \leq i \leq r_1. \end{cases}$$

It is easy to see that $\lambda(w_\varepsilon) \geq \lambda$ and $\mu(w_\varepsilon) \geq \mu$ (see 2.2 for the definition of $\lambda(w)$ and $\mu(w)$). This forces that $\lambda(w_\varepsilon) = \lambda$. It is easy to check that w_ε is in $\Gamma_\lambda \cap \Gamma_\lambda^{-1}$ and $\varepsilon(w_\varepsilon) = (1, 0, ..., 0)$.

Now Suppose that $\varepsilon_{ij} \geq 1$ and we have defined $w = w_\varepsilon \in \Gamma_\lambda \cap \Gamma_\lambda^{-1}$ such that
(1) $\varepsilon(w_\varepsilon) = \varepsilon$, here $\varepsilon = (\varepsilon_{11}, ..., \varepsilon_{1n_1}, ..., \varepsilon_{i1}, ..., \varepsilon_{i,j-1}, \varepsilon_{ij} - 1, 0, ..., 0)$,
(2) $w(a) = w_\lambda(a)$ if $a_{r_ij} < a \leq n$, and $w(a_{r_ij}) = w_\lambda(a_{r_ij})$ if $\varepsilon_{ij} - 1 = 0$.
(3) all $w(a) > 0$ whenever $a > 0$.
Then we define $u = w_{\varepsilon'}$ for $\varepsilon' = (\varepsilon_{11}, ..., \varepsilon_{1n_1}, ..., \varepsilon_{i1}, ..., \varepsilon_{i,j-1}, \varepsilon_{ij}, 0, ..., 0)$ as follows. Set $j_{r_i} = j$ and choose $1 \leq j_{k-1} \leq \lambda_{k-1}$ for $k = r_i, ..., 3, 2$ such that

$$w(a_{k-1,j_{k-1}-1}) > w(a_{k,j_k}) > w(a_{k-1,j_{k-1}})$$

for $k = r_i, ..., 3, 2$, (we set $w(a_{k-1,0}) = \infty$). Then define $u = w_{\varepsilon'}$ by

$$w_{\varepsilon'}(a) = \begin{cases} w(a_{k,j_k}) & \text{if } a = a_{k-1,j_{k-1}},\ 2 \leq k \leq r_i, \\ w(a_{1,j_1}) + n & \text{if } a = a_{r_i,j}, \\ w(a) & \text{if } a \not\equiv a_{kj_k} (\mathrm{mod}\ n),\ 1 \leq k \leq r_i. \end{cases}$$

Lemma 5.3.1. *$u = w_{\varepsilon'}$ is in $\Gamma_\lambda \cap \Gamma_\lambda^{-1}$ and $\varepsilon(u) = \varepsilon'$. Moreover, $u(a) = w_\lambda(a)$ if $a_{r_ij} < a \leq n$ and all $u(a) > 0$ whenever $a > 0$.*

Proof. We shall prove (1) $l(u) = l(uw_\lambda) + l(w_\lambda)$, (2) $l(u^{-1}) = l(u^{-1}w_\lambda) + l(w_\lambda)$, (3) $\lambda(u) = \lambda$, and (4) $\varepsilon(u) = \varepsilon'$.

Set

$$a_k = a_{kj_k} \quad \text{for } k = 1, 2, ..., r_i.$$

Write

$$w(a_k) = b_k + c_k n, \quad 1 \leq b_k \leq n,\ c_k \in \mathbb{Z}.$$

Then $b_k \in \Lambda_{q_k}$ for some $1 \leq q_k \leq r_i$, since by the assumption we have $w(\Lambda_q) = w_\lambda(\Lambda_q) = \Lambda_q$ if $r_i < q \leq r$. We have

$$w(a_{r_i,j-1}) = \xi + \eta n > b_{r_i} + c_{r_i} n = w(a_{r_ij})$$

since $w \in \Gamma_\lambda$, where $1 \leq \xi \leq n$ and $\eta \in \mathbb{Z}$.

It is easy to see that $w(a_1), ..., w(a_{r_i})$ form an r-antichain of w. Note that w is in $\Gamma_\lambda \cap \Gamma_\lambda^{-1}$. Since $w(\Lambda_q) = w_\lambda(\Lambda_q) = \Lambda_q$ whenever $r \geq q > r_i$, using Corollary 2.4.2 we see that each Λ_q $(1 \leq q \leq r_i)$ contains exactly one b_k $(1 \leq k \leq r_i)$. As in the proof of Lemma 5.1.4 we get

$(*)$ If b_{r_i} is in Λ_k, then (a) b_{r_i-h} is in Λ_{k-h} for all $1 \leq h < k$, and b_{r_i-k-h} is in Λ_{r_i-h} if $0 \leq h < r_i - k$; (b) $c_{r_i} = \cdots = c_{r_i-k+1}$, and $c_{r_i-k} = \cdots = c_1 = c_{r_i} - 1$.

(1) By the definition of u and using 2.1.3 (f), to check $l(u) = l(uw_\lambda) + l(w_\lambda)$, we only need to verify that $\xi + \eta n > b_1 + c_1 n + n = w(a_1) + n$.

When $j = 1$ we need do nothing. Now assume that $j > 1$, ξ is in Λ_k and b_{r_i} is in Λ_h. Note that $1 \leq k, h \leq r_i$.

If $k \leq h$, then $\eta > c_{r_i}$. Otherwise, $\varepsilon_{i,j-1}(w) \leq \varepsilon_{ij} - 1$. This contradicts that $\varepsilon_{i,j-1}(w) = \varepsilon_{i,j-1} \geq \varepsilon_{ij}$ since ε is in $\mathrm{Dom}(F_\lambda)$.

By $(*)$, we have $b_1 \in \Lambda_g$, where $g = h + 1$ if $1 \leq h < r_i$ and $g = 1$ if $h = r_i$.

If $1 \leq h < r_i$, then $c_1 = c_{r_i} - 1$. So $c_1 + 1 = c_{r_i} < \eta$. Thus we have $\xi + \eta n > w(a_1) + n$ in this case.

If $h = r_i$ and $k > 1$, then $g = 1$ and $c_1 = c_{r_i}$. In this case, $\xi > b_1$ and $\eta > c_1$, we also have $\xi + \eta n > w(a_1) + n$.

When $h = r_i,\ k = 1$, we have $g = 1$ and $c_1 = c_{r_i} < \eta$. We then have

$$w^{-1}(b_1) = a_1 - c_1 n > w^{-1}(\xi) = a_{r_i,j-1} - \eta n.$$

Since $w^{-1} \in \Gamma_\lambda$, we see $\xi > b_1$. Therefore $\xi + \eta n > w(a_1) + n$.

We have showed that if $k \leq h$ then $\xi + \eta n > w(a_1) + n$.

Now suppose that $k > h$. Then $\eta \geq c_{r_i}$ and $\xi > b_{r_i}$. We have $g = h + 1$ and $c_1 = c_{r_i} - 1$. If $k > h + 1$, we have $\xi > b_1$, so $\xi + \eta n > w(a_1) + n$. If $k = h + 1$, we have

$$w^{-1}(b_1) = a_1 - c_1 n > w^{-1}(\xi) = a_{r_i,j-1} - \eta n.$$

Since w^{-1} is in Γ_λ, thus $\xi > b_1$, so $\xi + \eta n > w(a_1) + n$. So if $k > h$ we also have $\xi + \eta n > w(a_1) + n$.

We have showed that $\xi + \eta n > w(a_1) + n$ is true. So $l(uw_\lambda) = l(u) - l(w_\lambda)$.

(2) Now we show that $l(u^{-1}w_\lambda) = l(u^{-1}) - l(w_\lambda)$. We have

$$u^{-1}(b) = \begin{cases} a_{r_i} - (c_1+1)n & \text{if } b = b_1, \\ a_{k-1} - c_k n & \text{if } b = b_k,\ 2 \leq k \leq r_i, \\ w^{-1}(b) & \text{if } b \not\equiv b_k (\text{mod } n),\ 1 \leq k \leq r_i. \end{cases}$$

Let $b > b'$ be in the same Λ_q for some q. We need to show that $u^{-1}(b) < u^{-1}(b')$. Note that $w^{-1}(b) < w^{-1}(b')$ since w^{-1} is in Γ_λ. When both b, b' are different from any b_k, we have

$$u^{-1}(b) = w^{-1}(b) < w^{-1}(b') = u^{-1}(b').$$

If $b = b_k > b'$ for some k, then $b' \neq b_h$ for any $1 \leq h \leq r_i$ since b, b' are in the same Λ_q and each Λ_q contains at most one element of $\{b_1, ..., b_{r_i}\}$. Then we have

$$u^{-1}(b) = u^{-1}(b_k) < w^{-1}(b_k) < w^{-1}(b') = u^{-1}(b').$$

Now suppose that $b > b_k = b'$. Then $b \neq b_h$ for any $1 \leq h \leq r_i$ since b, b' are in the same Λ_q and Λ_q contains at most one element of $\{b_1, ..., b_{r_i}\}$. Since $w^{-1} \in \Gamma_\lambda$ we have

$$w^{-1}(b_k) = a_k - c_k n > w^{-1}(b) = a - cn,$$

where $1 \leq a \leq n$ and $c \in \mathbb{Z}$. Thus $c_k \leq c$.

If $c_k < c$, we have

$$u^{-1}(b_k) = a_{k-1} - c_k n > a - cn = w^{-1}(b) = u^{-1}(b)$$

for $2 \leq k \leq r_i$. For $k = 1$, we have

$$u^{-1}(b_k) = u^{-1}(b_1) = a_{r_i} - (c_1 + 1)n.$$

We claim that $a_{r_i} > a$. Otherwise, $a > a_{r_i}$, then we must have $c = 0$ since $w(a) = w_\lambda(a)$ for $a_{r_i} = a_{r_i j} < a \leq n$ and then $c_1 < 0$ since $c_k = c_1 < c$. By assumptions on w, $c_1 \geq 0$, a contradiction, so $a_{r_i} > a$. Thus in this case we also have

$$u^{-1}(b_k) = u^{-1}(b_1) = a_{r_i} - c_1 n - n > a - cn = w^{-1}(b) = u^{-1}(b).$$

We have showed that $u^{-1}(b_k) > u^{-1}(b)$ if $c_k < c$.

If $c_k = c$, then $a_k > a$. Note that $a_k \in \Lambda_k$.

If $1 \le k < r_i$, we have

$$w(a_{k+1}) = b_{k+1} + c_{k+1}n > w(a) = b + cn > w(a_k) = b_k + c_k n,$$

since b, b_k are contained in the same Λ_q and b_{k+1} is not in Λ_q and $w(a_{k+1}) > w(a_k)$. By the definition of a_k, we see that a is not in Λ_k in this case. If $k = r_i$, we also have that a is not in Λ_k. Otherwise we have $a \in \Lambda_{r_i}$. Then $c > c_{r_i}$, since b, b_k are in the same Λ_q, $a_k > a$ and $\varepsilon_{i1} \ge \cdots \ge \varepsilon_{i,j-1} > \varepsilon_{ij} - 1$. This contradicts the assumption $c = c_k (= c_{r_i}$ in this case). Therefore in any case a is not in Λ_k.

Since $a_k > a$, we see that a is in Λ_h for some $1 \le h < k$. If $h + 1 < k$, then

$$u^{-1}(b_k) = a_{k-1} - c_k n > a - cn = u^{-1}(b).$$

If $h + 1 = k$, then a, a_{k-1} are contained in Λ_h. Since $b > b_k$ and $c = c_k$, we have

$$w(a) = b + cn > b_k + c_k n > b_{k-1} + c_{k-1} n = w(a_{k-1}).$$

Thus $a < a_{k-1}$ since $w \in \Gamma_\lambda$. Therefore

$$u^{-1}(b_k) = a_{k-1} - c_k n > a - cn = u^{-1}(b)$$

if $k = h + 1$.

We have showed that $l(u^{-1} w_\lambda) = l(u^{-1}) - l(w_\lambda)$.

(3) Now we prove that $\lambda(u) = \lambda$. By (1) we see $\lambda(u) \ge \lambda$. If we can show that $\mu(u) \ge \mu$, then we are forced to have $\lambda(u) = \lambda$.

Let Z be a complete r-antichain family of w. If $w(a_{r_i}) > n$, by the assumptions on w, we see that the r-antichain in Z that contains $w(a_{r_i})$ has length r_i.

If $w(a_{r_i}) < n$, then $\varepsilon_{ij}(w) = \varepsilon_{ij} - 1 = 0$. By the assumptions on w we see that $w(a_{r_i}) = w_\lambda(a_{r_i})$. Let B be the r-antichain in Z that provides $\varepsilon_{ij}(w)$ (i.e. $\varepsilon(B) = \varepsilon_{ij}(w)$ and B has length r_i). Let C be the r-antichain in Z that contains $w(a_{r_i})$. Assume that $w(a_{r_i l})$ is in B. Then by the assumptions on w, we have $l \ge j$ and $w(a_{r_i l}) \in \Lambda_{r_i}$. Replacing $w(a_{r_i l})$ in B by $w(a_{r_i})$ and replacing $w(a_{r_i})$ in C by $w(a_{r_i l})$, we get a complete r-antichain family Z' of w. The r-antichain in Z' that contains $w(a_{r_i})$ has length r_i.

Thus it is harmless to assume that the r-antichain in Z that contains $w(a_{r_i})$ has length r_i,

Let

$$I : w(\xi_s) > w(\xi_{s-1}) > \cdots > w(\xi_1),$$

$$J : w(\eta_t) > w(\eta_{t-1}) > \cdots > w(\eta_1)$$

be two r-antichains in Z. Note that ξ_k, η_k are contained in Λ_k. Assume that I contains $w(a_{r_i})$. Then $s = r_i$ and $\xi_{r_i} = a_{r_i}$. Write

$$w(\xi_k) = \alpha_k + \beta_k n, \quad w(\eta_k) = \delta_k + \theta_k n,$$

where $1 \le \alpha_k, \delta_k \le n$ and $\beta_k, \theta_k \in \mathbb{Z}$. Suppose that for some $1 \le h \le k < r_i$ we have

$$\begin{array}{l} w(\xi_{k+1}) > w(\eta_k) > w(\xi_k),\ w(\eta_{k-1}) > w(\xi_{k-1}), \\ ...,\ w(\eta_h) > w(\xi_h),\ w(\eta_{h-1}) < w(\xi_{h-1}). \end{array}$$

We claim that

(3a) α_k, δ_k are contained in the same Λ_q for some q and $\beta_k = \theta_k$.

Assume that α_k is in Λ_q. Since $w(\xi_{k+1}) > w(\eta_k) > w(\xi_k)$, we have $\beta_{k+1} \geq \theta_k \geq \beta_k$. If $1 \leq q \leq r_i - 1$, using Lemma 5.1.4, then $\alpha_{k+1} \in \Lambda_{q+1}$ and $\beta_{k+1} = \beta_k$. Thus $\alpha_{k+1} > \delta_k$ and δ_k is in Λ_q or Λ_{q+1}. In this case we have

$$w^{-1}(\delta_k) = \eta_k - \theta_k n = \eta_k - \beta_k n < \xi_{k+1} - \beta_k n = w^{-1}(\alpha_{k+1}).$$

Since $w^{-1} \in \Gamma_\lambda$, we must have $\delta_k \in \Lambda_q$. Thus (3a) is true if $1 \leq q \leq r_i - 1$.

Now suppose that $q = r_i$. By Lemma 5.1.4, $\alpha_{k+1} \in \Lambda_1$ and $\beta_{k+1} = \beta_k + 1$. Thus $\theta_k = \beta_k$ or $\beta_k + 1$. If $\theta_k = \beta_k + 1$, then we must have $\delta_k \in \Lambda_1$ and $\alpha_{k+1} > \delta_k$, since $w(\xi_{k+1}) > w(\eta_k) > w(\xi_k)$. But in this case we have $w^{-1}(\delta_k) < w^{-1}(\alpha_{k+1})$. This contradicts that $w^{-1} \in \Gamma_\lambda$. Therefore $\theta_k = \beta_k$. Now $1 \leq k \leq r_i - 1$ and by assumptions on w we have $w(\Lambda_l) = w_\lambda(\Lambda_l) = \Lambda_l$ if $r_i < l \leq r$. Thus δ_k is not in Λ_l for any $r_i < l \leq r$. This forces that $\delta_k \in \Lambda_{r_i} = \Lambda_q$ since $w(\eta_k) > w(\xi_k)$ and $\theta_k = \beta_k$. In conclusion, (3a) is true if $q = r_i$.

We have seen that (3a) is always true.

Thus the following two sequences

$$I' : w(\xi_s) > \cdots > w(\xi_{k+1}) > w(\eta_k) > \cdots > w(\eta_h)$$

$$> \cdots > w(\xi_{h-1}) > \cdots > w(\xi_1),$$

$$J' : w(\eta_t) > \cdots > w(\eta_{k+1}) > w(\xi_k) > \cdots > w(\xi_h)$$

$$> \cdots > w(\eta_{h-1}) > \cdots > w(\eta_1)$$

are two r-antichains of w. Replacing I, J in Z by I', J' respectively, we get a new complete r-antichain family of w. Continuing this process, finally we can find a complete r-antichain family Y of w with the following two properties

(3b) The r-antichain I in Y that contains $w(a_{r_i})$ has length r_i,

(3c) Let $I : w(\xi_s) > w(\xi_{s-1}) > \cdots > w(\xi_1)$, $J : w(\eta_t) > w(\eta_{t-1}) > \cdots > w(\eta_1)$, be two r-antichains in Y with $s = r_i$ and $\xi_{r_i} = a_{r_i}$. For any $1 \leq k \leq s-1, t$, if $w(\xi_{k+1}) > w(\eta_k)$, then $w(\xi_k) > w(\eta_k)$.

By the definition of a_k, we see $\xi_k = a_k$.

Replacing the r-antichain $w(a_{r_i}) > \cdots > w(a_2) > w(a_1)$ in Y by $u(a_{r_i}) > \cdots > u(a_2) > u(a_1)$, we then get a set Y' consisting of r' r-antichains of u. The lengths of the r-antichains of u in Y' are $\mu_1, ..., \mu_{r'}$ respectively. Moreover the union of all elements in the r-antichains of u in Y' is $\{u(1), u(2), ..., u(n)\}$. By definition we have $\mu(u) \geq \mu$. Therefore we have $\mu(u) = \mu$ and $\lambda(u) = \lambda$.

(4) From the proof above we see clearly $\varepsilon(u) = \varepsilon'$.

From the definition of u we see $u(a) = w_\lambda(a)$ if $a_{r_i j} < a \leq n$ and all $u(a) > 0$ whenever $a > 0$.

The lemma is proved.

Lemma 5.3.2. *Let w be in $\Gamma_\lambda \cap \Gamma_\lambda^{-1}$. If all $\varepsilon_{ij}(w)$ are non-negative, then $w(k) > 0$ for all $1 \leq k \leq n$.*

Proof. Otherwise, we can find some $1 \leq k \leq n$ such that $w(k) = a - bn$, where $1 \leq a \leq n$ and b is a positive integer. Let I be an r-antichain of w in a complete r-antichain family of w that contains $w(k)$. Let $c + dn$ ($1 \leq c \leq n,\ d \in \mathbb{Z}$)

be the largest number in I. Since $\varepsilon_{ij}(w) \geq 0$ for all i, j, we have $d \geq 1$. Then $c + dn > a - bn + n$. This contradicts that I is an r-antichain of w containing $w(k)$. The lemma is proved.

Lemma 5.3.3. *Let $w \in \Gamma_\lambda \cap \Gamma_\lambda^{-1}$ be such that all components of $\varepsilon_{ij}(w)$ are non-negative. Let $1 \leq i \leq p$ and $1 \leq j \leq n_i$. If all $\varepsilon_{kl}(w) = 0$ for $k > i$, or $k = i$ and $l > j$, then $w(a) = w_\lambda(a)$ whenever $a_{r_i j} < a \leq n$.*

Proof. By Lemma 5.3.2, w is positive. By the assumption on w we see that $w(a) \leq n$ if $a_{r_i j} < a \leq n$. Using Lemma 5.1.7 we see that $w(a) = w_\lambda(a)$ if $a_{r_i j} < a \leq n$. The lemma is proved.

5.3.4. Let w be in $\Gamma_\lambda \cap \Gamma_\lambda^{-1}$. Suppose that $\varepsilon_{ij} \geq 1$ and all ε_{kl} are non-negative and we have

$$\varepsilon(w) = (\varepsilon_{11}, ..., \varepsilon_{1n_1}, ..., \varepsilon_{i1}, ..., \varepsilon_{i,j-1}, \varepsilon_{ij}, 0, ..., 0).$$

Then we define u for

$$\varepsilon' = (\varepsilon_{11}, ..., \varepsilon_{1n_1}, ..., \varepsilon_{i1}, ..., \varepsilon_{i,j-1}, \varepsilon_{ij} - 1, 0, ..., 0)$$

as follows. Choose $1 \leq j_1 \leq \lambda_1$ such that $w(a_{1j_1}) > w(a_{r_i j}) - n$ but $w(a_{1,j_1+1}) < w(a_{r_i j}) - n$ if $j_1 + 1 \leq \lambda_1$. Then choose $2 \leq j_k \leq \lambda_k$ for $k = 1, 2, ..., r_i - 1$ such that

$$w(a_{k,j_k}) > w(a_{k-1,j_{k-1}}) > w(a_{k,j_k+1})$$

for $k = 2, 3, ..., r_i - 1$. (We set $w(a_{k,\lambda_k+1}) = -\infty$ for all k). By Lemma 5.1.8, such $j_1, ..., j_{r_i-1}$ exist. Finally let $j_{r_i} = j$. For simplicity we set

$$a_k = a_{kj_k} \quad \text{for } k = 1, ..., r_i.$$

Now we define $u = u_{\varepsilon'}$ by

$$u(a) = \begin{cases} w(a_{r_i}) - n & \text{if } a = a_1, \\ w(a_{k-1}) & \text{if } a = a_k,\ 2 \leq k \leq r_i, \\ w(a) & \text{if } a \not\equiv a_k (\text{mod } n), \quad \text{for all } 1 \leq k \leq r_i. \end{cases}$$

Actually we have defined this element in section 5.1 (see Lemma 5.1.11). According to Lemma 5.1.11 we get

5.3.4 (a) u is in $\Gamma_\lambda \cap \Gamma_\lambda^{-1}$.

From the proofs of Lemma 5.1.11 and of Theorem 5.1.2 we see clearly

5.3.4 (b) $\varepsilon(u) = \varepsilon' = (\varepsilon_{11}, ..., \varepsilon_{1n_1}, ..., \varepsilon_{i1}, ..., \varepsilon_{i,j-1}, \varepsilon_{ij} - 1, 0, ..., 0)$.

5.3.5. *Proof of Theorem 5.2.6 (b):* Let $\varepsilon = (\varepsilon_{11}, ..., \varepsilon_{1n_1}, ..., \varepsilon_{p1}, ..., \varepsilon_{pn_p})$ be in $\text{Dom}(F_\lambda)$. Choose $k \in \mathbb{N}$ such that $\varepsilon_{ij} + kr_i \geq 0$ for all $1 \leq i \leq p,\ 1 \leq j \leq n_i$. By Lemma 5.3.1 we can find $w \in \Gamma_\lambda \cap \Gamma_\lambda^{-1}$ such that

$$\varepsilon(w) = (\varepsilon_{11} + kr_1, ..., \varepsilon_{1n_1} + kr_1, ..., \varepsilon_{p1} + kr_p, ..., \varepsilon_{pn_p} + kr_p) \in \text{Dom}(F_\lambda).$$

Then we have $\omega^{-kn} w \in \Gamma_\lambda \cap \Gamma_\lambda^{-1}$ and

$$\varepsilon(\omega^{-kn} w) = (\varepsilon_{11}, ..., \varepsilon_{1n_1}, ..., \varepsilon_{p1}, ..., \varepsilon_{pn_p}).$$

Therefore the map ε is surjective.

Suppose that $\varepsilon(w) = (0, ..., 0) \in$Dom$(F_\lambda)$. By Lemma 5.3.2, w is positive. By the definition of $\varepsilon(w)$, we see $w(a) \le n$ if $1 \le a \le n$. Thus $w(1) + \cdots + w(n) = 1 + \cdots + n$. Using Lemma 5.1.7 we get $w = w_\lambda$. Now suppose that $w, w' \in \Gamma_\lambda \cap \Gamma_\lambda^{-1}$ and

$$\varepsilon(w) = \varepsilon(w') = \varepsilon = (\varepsilon_{11}, ..., \varepsilon_{1n_1}, ..., \varepsilon_{i1}, ..., \varepsilon_{ij}, 0..., 0).$$

Suppose all ε_{kl} are nonnegative and $\varepsilon_{ij} \ge 1$. By subsection 5.3.4 we can construct u, u' from w, w' respectively such that

$$\varepsilon(u) = \varepsilon(u') = \varepsilon = (\varepsilon_{11}, ..., \varepsilon_{1n_1}, ..., \varepsilon_{i,j-1}, \varepsilon_{ij} - 1, 0..., 0).$$

We use induction on the sum of all components of $\varepsilon(w)$. By induction hypothesis we see $u = u'$. Now we can recover w, w' from u, u' using the construction at the beginning of this section, see Lemma 5.3.1. Therefore $w = w'$ if all components of $\varepsilon(w) = \varepsilon(w')$ are nonnegative. In general we can find k such that all components of $\varepsilon(\omega^{kn} w) = \varepsilon(\omega^{kn} w')$ are nonnegative. Thus $\omega^{kn} w = \omega^{kn} w$. Hence $w = w'$ if $\varepsilon(w) = \varepsilon(w')$. We proved that the map ε is injective.

Theorem 5.2.6 (b) is proved.

In the following section we give some simple properties of elements in $\Gamma_\lambda \cap \Gamma_\lambda^{-1}$.

5.4. Some simple properties of elements in $\Gamma_\lambda \cap \Gamma_\lambda^{-1}$

Recall that we defined $a_{ij} = \lambda_1 + \cdots + \lambda_{i-1} + j$ for $1 \le i \le r$, $1 \le j \le \lambda_i$, and $\Lambda_k = \{a_{k1}, ..., a_{k\lambda_k}\}$. Fix w in $\Gamma_\lambda \cap \Gamma_\lambda^{-1}$. We write $w(a_{ij}) = b_{ij} + l_{ij} n$, where $1 \le b_{ij} \le n$ and $l_{ij} \in \mathbb{Z}$.

Lemma 5.4.1. *Assume that $1 \le j < k \le \lambda_i$. If both b_{ij} and b_{ik} are contained in the same Λ_q for some q, then $b_{ij} > b_{ik}$.*

Proof. Note that $l_{ij} \ge l_{ik}$ since $w(a_{ij}) > w(b_{ik})$. Thus we have

$$w^{-1}(b_{ij}) = a_{ij} - l_{ij} n < w^{-1}(b_{ik}) = a_{ik} - l_{ik} n.$$

Since $w^{-1} \in \Gamma_\lambda$ and both b_{ij} and b_{ik} are contained in Λ_q, we see $b_{ij} > b_{ik}$.

Lemma 5.4.2. *Assume that $1 \le a, b \le n$ and $w(a) = a_{ij} + l_a n$ and $w(b) = a_{ik} + l_b n$, where $1 \le j < k \le \lambda_{i+1}$. Then $l_a \le l_b$. Moreover, if $l_a = l_b$ then $a > b$.*

Proof. Since $w^{-1}(a_{ij}) > w^{-1}(a_{ik})$ we see $l_a \le l_b$. If $l_a = l_b$, from $w^{-1}(a_{ij}) > w^{-1}(a_{ik})$ we get $a > b$.

Lemma 5.4.3. (a) $w(\lambda_1) = e_i + 1 (\mathrm{mod}\ n)$ *for some* i.
(b) $w(e_i + 1) \equiv \lambda_1 (\mathrm{mod}\ n)$ *for some* i.
(c) $w(1) \equiv e_i (\mathrm{mod}\ n)$ *for some* i.
(d) $w(e_i) \equiv 1 (\mathrm{mod}\ n)$ *for some* i.
(e) *If $w(a_{i1})$ is maximal among all $w(a_{jk})$, then $w(a_{i1}) \equiv e_j (\mathrm{mod}\ n)$ for some j.*

Proof. (a) Assume that $w(\lambda_1) = e_i + j + l_{\lambda_1} n$, where $1 \le j \le \lambda_{i+1}$ and $l_{\lambda_1} \in \mathbb{Z}$. If $j \ne 1$, we can find $1 \le k \le n$ such that $w(k) = e_i + 1 + l_k n$. Then $l_k > l_{\lambda_1}$ since $w(k) > w(\lambda_1)$, see Lemma 5.1.1 (b). Thus $w^{-1}(e_i + 1) < w^{-1}(e_i + j)$. This contradicts $w^{-1} \in \Gamma_\lambda$. So we must have $j = 1$.

(b) Applying (a) to w^{-1} we see that (b) is true.

(c) Assume that $w(1) = e_{i-1} + j + l_1 n$, where $1 \leq j \leq \lambda_i$ and $l_1 \in \mathbb{Z}$. If $j \neq \lambda_i$, we can find $1 \leq k \leq n$ such that $w(k) = e_i + l_k n$. Since $w^{-1} \in \Gamma_\lambda$, we have $l_1 < l_k$. Then $w(k) > w(1+n) > \cdots > w(\lambda_1 + n)$. This contradicts $\lambda(w) = \lambda$. So $j = \lambda_i$.

(d) It follows from $w^{-1} \in \Gamma_\lambda \cap \Gamma_\lambda^{-1}$ and (c).

The proof of (e) is similar to that of (a).

Lemma 5.4.4. *Let $w \in \Gamma_\lambda \cap \Gamma_\lambda^{-1}$ be such that all components of $\varepsilon(w)$ are non-negative. If the (k,l)-component of $\varepsilon(w)$ is 0 whenever $k \leq i-1$, then $w(a_{\alpha\gamma}) > w(a_{\beta\gamma})$ whenever $r_i \geq \alpha > \beta \geq 1$ and $\lambda_\alpha, \lambda_\beta \geq \gamma \geq 1$.*

Proof. It follows from the construction of elements in $\Gamma_\lambda \cap \Gamma_\lambda^{-1}$, see section 5.3. To see it more clearly we use induction on the sum of all components of $\varepsilon(w)$ as in section 5.3. When the sum of all components of $\varepsilon(w)$ is 0, we have $w = w_\lambda$. In this case the lemma is true. Now assume that

$$\varepsilon(w) = (0, ..., 0, \varepsilon_{i1}(w), ..., \varepsilon_{in_i}(w), ..., \varepsilon_{j1}(w), ..., \varepsilon_{jh}(w), 0, ..., 0) \in \mathrm{Dom}(F_\lambda),$$

where $i \leq j$, $1 \leq h \leq n_j$ and $\varepsilon_{jh}(w) \geq 1$.

Let u be in $\Gamma_\lambda \cap \Gamma_\lambda^{-1}$ such that $\varepsilon_{\alpha\beta}(w) = \varepsilon_{\alpha\beta}(w)$ whenever $(\alpha, \beta) \neq (j,h)$ and $\varepsilon_{jh}(u) = \varepsilon_{jh}(w) - 1$. By induction hypothesis, we have

$$(*) \qquad u(a_{\alpha\gamma}) > u(a_{\beta\gamma})$$

whenever $r_i \geq \alpha > \beta \geq 1$ and $\lambda_\alpha, \lambda_\beta \geq \gamma \geq 1$.

According to Lemma 5.3.1 and Theorem 5.2.6, we have

$$w(a) = \begin{cases} u(a_{k,m_k}) & \text{if } a = a_{k-1,m_{k-1}},\ 2 \leq k \leq r_j, \\ u(a_{1,m_1}) + n & \text{if } a = a_{r_j,h}, \\ w(a) & \text{if } a \not\equiv a_{km_k} (\text{mod } n),\ 1 \leq k \leq r_j, \end{cases}$$

where $m_{r_j} = h$ and $m_{r_j - 1}, ..., m_2, m_1$ are inductively defined by

$$w(a_{k,m_k - 1}) > w(a_{k+1,m_{k+1}}) > w(a_{k,m_k})$$

for $k = r_j - 1, ..., 2, 1$, (we set $w(a_{k,0}) = \infty$). Using $(*)$ we see that

$(\star)$ $m_{r_j} \geq m_{r_j - 1} \geq \cdots \geq m_2 \geq m_1$.

Let $r_i \geq \alpha > \beta \geq 1$ and $\lambda_\alpha, \lambda_\beta \geq \gamma \geq 1$.

If both (α, γ) and (β, γ) are different from any (k, m_k) for $k = 1, 2, ..., r_j$, we have

$$w(a_{\alpha\gamma}) = u(a_{\alpha\gamma}) > u(a_{\beta\gamma}) = w(a_{\beta\gamma}).$$

If $(\alpha, \gamma) = (q, m_q)$ for some $1 \leq q \leq r_j$ and (β, γ) is different from any (k, m_k) for $k = 1, 2, ..., r_j$, noting that $u(a_{1m_1})$,...,$u(a_{r_j m_{r_j}})$ form an r-antichain of u (see the proof of Lemma 5.3.1), we have

$$w(a_{\alpha\gamma}) > u(a_{\alpha\gamma}) > u(a_{\beta\gamma}) = w(a_{\beta\gamma}).$$

Suppose that (α, γ) is different from any (k, m_k) for $k = 1, 2, ..., r_i$ and $(\beta, \gamma) = (q, m_q)$ for some $1 \leq q \leq r_i$. By $(\star)$ we have $m_\alpha \geq \cdots \geq m_{q+1} \geq m_q = \gamma$. Now $\gamma \neq m_\alpha$, so $\gamma < m_\alpha$. When $\alpha = \beta + 1$, we then have

$$w(a_{\alpha\gamma}) = u(a_{\alpha\gamma}) > u(a_{\alpha,\gamma+1}) \geq u(a_{\alpha m_\alpha}) = w(a_{\beta\gamma}).$$

When $\alpha > \beta + 1$, we have

$$w(a_{\alpha\gamma}) = u(a_{\alpha\gamma}) > u(a_{\alpha,m_\alpha}) > \cdots > u(a_{\beta+1,m_{\beta+1}}) = w(a_{\beta\gamma}).$$

Finally suppose that $(\alpha, \gamma) = (\alpha, m_\alpha)$ and $(\beta, \gamma) = (\beta, m_\beta)$. By the relation between w and u we have

$$w(a_{\alpha\gamma}) > w(a_{\beta\gamma}).$$

By induction we see that the lemma is true.

5.5. Some elements of $\Gamma_\lambda \cap \Gamma_\lambda^{-1}$

In this section we write explicitly the elements in $\Gamma_\lambda \cap \Gamma_\lambda^{-1}$ corresponding to fundamental weights of F_λ.

Let θ_{ij} $(1 \le i \le p,\ 1 \le j \le n_i)$ be in $\mathrm{Dom}(F_\lambda)$ such that its (k,l)-component is 0 whenever $k \ne i$ or $l > j$ and its (i,l)-component is 1 for $l = 1, ..., j$. Then θ_{ij} is the (i,j)th fundamental weight of F_λ. Given $1 \le i_1, ..., i_k \le n$, let $s(i_1, ..., i_k)$ be the element of W defined by $i_l \to i_{l+1}$ for $l = 1, ..., k-1$, $i_l \to i_1$, and $m \to m$ if $m \not\equiv i_l (\mathrm{mod} n)$ for all l.

Let $1 \le i \le p$ and $1 \le j \le n_i$. For simplicity we write h for r_i. Define

$$u_{ij} = \tau_{\lambda_1} s(e_1, ..., e_h) \tau_{\lambda_1 - 1} s(e_1 - 1, ..., e_h - 1) \cdots \tau_{\lambda_1 - j + 1} s(e_1 - j + 1, ..., e_h - j + 1).$$

Then

$$u_{ij}(a) = \begin{cases} e_{k+1} - l + 1, & \text{if } a = e_k - l + 1, \quad 1 \le k \le h-1, \quad 1 \le l \le j; \\ e_1 - l + 1 + n, & \text{if } a = e_h - l + 1, \quad 1 \le l \le j; \\ a, & \text{if } a \ne e_k - l + 1, \quad \text{for all } 1 \le k \le h, \quad 1 \le l \le j, \end{cases}$$

where $1 \le a \le n$. Note that $w_\lambda(e_{k-1} + l) = e_k - l + 1$ for all $1 \le k \le r$ and $1 \le l \le \lambda_k$. Thus we have

$$u_{ij} w_\lambda(a) = \begin{cases} e_{k+1} - l + 1, & \text{if } a = e_{k-1} + l, \quad 1 \le k \le h-1, \quad 1 \le l \le j; \\ e_1 - l + 1 + n, & \text{if } a = e_{h-1} + l, \quad 1 \le l \le j; \\ w_\lambda(a), & \text{if } a \ne e_{k-1} + l, \quad \text{for all } 1 \le k \le h, \quad 1 \le l \le j, \end{cases}$$

where $1 \le a \le n$.

Recall that $a_{kl} = e_{k-1} + l$ for $1 \le k \le r$ and $1 \le l \le \lambda_k$. According to Lemma 5.3.1 we see easily

(a) $u_{ij} w_\lambda$ is in $\Gamma_\lambda \cap \Gamma_\lambda^{-1}$.

(b) $\varepsilon(u_{ij} w_\lambda) = \theta_{ij}$.

Using 2.1.3 (e) and noting that $u_{ij} s_k \ge u_{ij}$ if $e_{m-1} + 1 \le k \le e_m - 1$ for some $1 \le m \le r$ (cf. 2.1.3 (f)), we get

(c) $l(u_{ij}) = (n - r_i j) j$ and $l(u_{ij} w_\lambda) = l(u_{ij}) + l(w_\lambda)$.

Let $j\theta_{i1} \in \mathrm{Dom}(F_\lambda)$ be such that its $(i,1)$-component is j and other components are 0. Using Lemma 5.3.1 we see

(d) $u_{i1}^j w_\lambda$ is in $\Gamma_\lambda \cap \Gamma_\lambda^{-1}$.

(e) $\varepsilon(u_{i1}^j w_\lambda) = j\theta_{i1}$.

Using 2.1.3 (e) we see that

$$l(u_{i1}^{jr_i}) = (n - r_i) j r_i = j r_i l(u_{i1}).$$

Obviously we have $u_{i1}^j s_k \geq u_{i1}^j$ if $e_{m-1}+1 \leq k \leq e_m - 1$ for some $1 \leq m \leq r$ (cf. 2.1.3 (f)). Therefore we have

(f) $l(u_{i1}^j) = jl(u_{i1})$ and $l(u_{i1}^j w_\lambda) = jl(u_{i1}) + l(w_\lambda)$.

Using the reduced expressions in 2.1.3 (d) we get

(g) A reduced expression of u_{i1} is

$$\begin{aligned} &\omega s_{e_1-2} s_{e_1-3} \cdots s_1 s_0 s_{n-1} s_{n-2} \cdots s_{e_h} \\ &\times \hat{s}_{e_h-1} s_{e_h-2} \cdots s_{e_{h-1}} \hat{s}_{e_{h-1}-1} s_{e_{h-1}-2} \cdots s_{e_{h-2}} \\ &\times \cdots \hat{s}_{e_2-1} s_{e_2-2} \cdots s_{e_1} \end{aligned}$$

where $\hat{s}_k$ means that s_k is omitted.

CHAPTER 6

A Factorization Formula in $J_{\Gamma_\lambda\cap\Gamma_\lambda^{-1}}$

In this chapter we will establish a factorization formula in $J_{\Gamma_\lambda\cap\Gamma_\lambda^{-1}}$, see Theorem 6.4.1. This formula is important for our proof of Lusztig Conjecture on based ring for type $\tilde{A}_{n-1}$ and is obviously motivated by the corresponding formula in R_{F_λ}, which says that each irreducible rational F_λ-module is a tensor product of some irreducible modules of reductive components of F_λ. In section 6.1 we note that in some cases the integers $\gamma_{u,v,w}$ can be computed through the basis $\{\tilde{T}_x \mid x\in W\}$ instead of the basis $\{C_x \mid x\in W\}$. In section 6.2 we compute the product $\tilde{T}_u\tilde{T}_v$ for some special u,v. This computation is a key to our factorization formula. In section 6.3 we give some consequences of the computation in section 2. In section 6.4 we prove the factorization formula.

6.1. The integers $\gamma_{u,v,w}$

In this section we show that in some cases the integer $\gamma_{u,v,w}$ can be computed through the product $\tilde{T}_u\tilde{T}_v$ instead of C_uC_v, the latter is usually much more difficult to compute. Let λ and w_λ be as in section 2.2. Using induction on $l(w)$ we see

(a) If $w=uw_\lambda$ and $l(w)=l(u)+l(w_\lambda)$, then $C_w=\phi C_{w_\lambda}$ for some $\phi\in H$ with $\bar{\phi}=\phi$.

(b) If $w=w_\lambda u$ and $l(w)=l(u)+l(w_\lambda)$, then $C_w=C_{w_\lambda}\phi$ for some $\phi\in H$ with $\bar{\phi}=\phi$.

Noting that if u,v are in $\Gamma_\lambda\cap\Gamma_\lambda^{-1}$, we can find u_1,v_1 in W such that $u=u_1w_\lambda$, $v=w_\lambda v_1$ and $l(u)=l(u_1)+l(w_\lambda)$, $l(v)=l(v_1)+l(w_\lambda)$. Thus we can find bar invariant elements $\phi_1,\phi_2\in H$ such that

$$C_u=\phi_1C_{w_\lambda}\ \text{and}\quad C_v=C_{w_\lambda}\phi_2.$$

Note that

$$C_{w_\lambda}C_{w_\lambda}=q^{-l(w_\lambda)}\sum_{w\le w_\lambda}q^{2l(w)}C_{w_\lambda}.$$

Let

$$\phi_1C_{w_\lambda}\phi_2=\sum_{w\in W}\eta_wC_w,\ \eta_w\in\mathcal{A}.$$

Then η_w is bar invariant, i.e. $\bar{\eta}_w=\eta_w$. We have

$$h_{u,v,w}=q^{-l(w_\lambda)}\sum_{y\le w_\lambda}q^{2l(y)}\eta_w$$

and then the degree of $h_{u,v,w}$ is either bigger than or equal to $a(w_\lambda)$ whenever $\eta_w\neq 0$. If w is in $\Gamma_\lambda\cap\Gamma_\lambda^{-1}$, we must have $\eta_w\in\mathbb{N}$ and $\eta_w=\gamma_{u,v,w}$.

Lemma 6.1.1. *Let $u, v \in \Gamma_\lambda \cap \Gamma_\lambda^{-1}$. Write (see 1.2 for the definition of $\tilde{T}_w$)*

$$\tilde{T}_u\tilde{T}_v = \sum_{w\in W} f_{u,v,w}\tilde{T}_w, \quad f_{u,v,w} \in \mathcal{A}.$$

If $\deg f_{u,v,w} \leq a(w_\lambda) = a$ for all w, then

$$f_{u,v,w} = \gamma_{u,v,w}q^a + \textit{lower degree terms},$$

for any $w \in \Gamma_\lambda \cap \Gamma_\lambda^{-1}$.

Proof. Write

$$\tilde{T}_u\tilde{T}_v = \sum_{w\in W} h'_{u,v,w}C_w, \ h'_{u,v,w} \in \mathcal{A}.$$

Note that

$$\tilde{T}_x \in C_x + q^{-1}\sum_{y\in W}\mathbb{Z}[q^{-1}]C_y \quad \text{ for any } x \in W.$$

We see $\deg h'_{u,v,w} \leq a(w_\lambda) = a$ for any w since $\deg f_{u,v,w} \leq a$. Moreover $\deg h'_{u,v,w} = a$ if and only if $\deg f_{u,v,w} = a$, and in this case the leading coefficients of $h'_{u,v,w}$ and $f_{u,v,w}$ coincide.

For $w \in \Gamma_\lambda \cap \Gamma_\lambda^{-1}$ we have

$$h'_{u,v,w} = \gamma_{u,v,w}q^a + \text{ lower degree terms.}$$

Therefore

$$f_{u,v,w} = \gamma_{u,v,w}q^a + \text{lower degree terms},$$

for any $w \in \Gamma_\lambda \cap \Gamma_\lambda^{-1}$.

The lemma is proved.

Lemma 6.1.2. *Let $u, v \in W$ and $s_{i_1}\cdots s_{i_k}$ a reduced expression of v. Then*

$$\tilde{T}_u\tilde{T}_v = \sum_{\mathbf{j}} g_{u,v,\mathbf{j}}\tilde{T}_{uw_{\mathbf{j}}}, \quad g_{u,v,\mathbf{j}} \in \mathcal{A},$$

where $\mathbf{j}$ runs through all subsequences $j_1, ..., j_m$ (including empty subsequence) of $i_1, ..., i_k$ and $w_{\mathbf{j}} = s_{j_1}\cdots s_{j_m}$.

Proof. It follows from the following fact

$$\tilde{T}_x\tilde{T}_{s_i} = \begin{cases} (q-q^{-1})\tilde{T}_x + \tilde{T}_{xs_i} & \text{if } xs_i \leq x \\ \tilde{T}_{xs_i} & \text{if } xs_i \geq x. \end{cases}$$

Obviously when $i_1, ..., i_k$ are pairwise different, then $w_{\mathbf{j}} \neq w_{\mathbf{j}'}$ if $\mathbf{j} \neq \mathbf{j}'$, and $g_{u,v,\mathbf{j}}$ is either 0 or a power of $q - q^{-1}$.

Lemma 6.1.3. *Let $x \in W$ and j, k be integers with $0 \leq k - j \leq n - 2$. Assume that $xs_l \leq x$ for $l = j, j+1, ..., k$, and let $y = s_ks_{k-1}\cdots s_j$. Then*

$$\tilde{T}_x\tilde{T}_y = \sum_{k\geq k_1>k_2>\cdots>k_m\geq j} \xi^{k-j+1-m}\tilde{T}_{xk_1\cdots k_m},$$

here we use $xk_1\cdots k_m$ for $xs_{k_1}\cdots s_{k_m}$, and ξ stands for $q - q^{-1}$.

Proof. We use induction on $k - j$. When $k - j = 0$, we have

$$\tilde{T}_x\tilde{T}_y = \xi\tilde{T}_x + \tilde{T}_{xs_k}.$$

The lemma is true in this case. Now suppose that the lemma is true when y is replaced by ys_j. Then we have

$$\tilde{T}_x\tilde{T}_{ys_j} = \sum_{k\ge k_1>k_2>\cdots>k_m\ge j+1} \xi^{k-j-m}\tilde{T}_{xk_1\cdots k_m}.$$

Note that $xs_{k_1}\cdots s_{k_m}s_j \le xs_{k_1}\cdots s_{k_m}$ for any sequence $k\ge k_1>k_2>\cdots>k_m\ge j+1$. Thus we have

$$\begin{aligned}\tilde{T}_x\tilde{T}_y &= \tilde{T}_x\tilde{T}_{ys_j}\tilde{T}_{s_j}\\ &= \textstyle\sum_{k\ge k_1>k_2>\cdots>k_m\ge j+1}\xi^{k-j-m}\tilde{T}_{xk_1\cdots k_m}\tilde{T}_{s_j}\\ &= \textstyle\sum_{k\ge k_1>k_2>\cdots>k_m\ge j+1}\xi^{k-j-m}(\xi\tilde{T}_{xk_1\cdots k_m}+\tilde{T}_{xk_1\cdots k_mj})\\ &= \textstyle\sum_{k\ge k_1>k_2>\cdots>k_m\ge j}\xi^{k-j+1-m}\tilde{T}_{xk_1\cdots k_m}.\end{aligned}$$

The lemma is proved.

Let W_λ be the subgroup of W generated by all simple reflections that appear in a reduced expression of w_λ.

Lemma 6.1.4. *Let $u\in W$. Write $u=u_1u_2$ such that u_1 is the shortest element in the coset uW_λ and u_2 is in W_λ. Then in*

$$\tilde{T}_u\tilde{T}_{w_\lambda} = \sum_{w\in W} f_{u,w_\lambda,w}\tilde{T}_w,\quad f_{u,w_\lambda,w}\in\mathcal{A},$$

we have $\deg f_{u,w_\lambda,w} < l(u_2)$ if $w\ne u_1w_\lambda$ and $f_{u,w_\lambda,u_1w_\lambda} = q^{l(u_2)}+$ lower degree terms.

Proof. We have $l(u_1)+l(u_2)=l(u)$, $l(u_1w_\lambda)=l(u_1)+l(w_\lambda)$ and $l(u_2w_\lambda)=l(w_\lambda)-l(u_2)$. Using induction on $l(u_2)$ we can prove the lemma.

In next section we compute some $\tilde{T}_u\tilde{T}_v$.

6.2. A computation for some $\tilde{T}_u\tilde{T}_v$

In this section we compute $\tilde{T}_u\tilde{T}_v$ for some special u,v in $\Gamma_\lambda\cap\Gamma_\lambda^{-1}$. Suppose that $w\in\Gamma_\lambda\cap\Gamma_\lambda^{-1}$ and $\varepsilon(w)=(\varepsilon_{11}(w),...,\varepsilon_{pn_p}(w))$. Denote by $\varepsilon_i(w)$ the element in $\mathrm{Dom}(F_\lambda)$ whose (i,j)-component is $\varepsilon_{ij}(w)$ for $j=1,2,...,n_i$ and other components are 0. Then $\varepsilon(w)=\varepsilon_1(w)+\cdots+\varepsilon_p(w)$. The main result of this section is

Proposition 6.2.1. *Let $u,v\in\Gamma_\lambda\cap\Gamma_\lambda^{-1}$ be such that all components of $\varepsilon(u)$ and $\varepsilon(v)$ are nonnegative. Assume that $\varepsilon_1(u)=\varepsilon_2(u)=\cdots=\varepsilon_{i-1}(u)=0$ and $\varepsilon_{i1}(v)=1$ and $\varepsilon_{kl}(v)=0$ if $(k,l)\ne(i,1)$. Then $\deg f_{u,v,w}\le a(w_\lambda)$ for all $w\in W$, see Lemma 6.1.1 for the definition of $f_{u,v,w}$.*

6.2.2. *Proof of Proposition 6.2.1.* When $r=1$, the proposition is proved in [L1]. Now we suppose that $r>1$. Recall that $e_0=0$, $e_k=\lambda_1+\lambda_2+\cdots+\lambda_k$, $a_{kl}=e_{k-1}+l$ for $k=1,2,...,r$, $1\le l\le\lambda_k$. Set $h=r_i$, see section 5.1 for definition of r_i. Let

$$\begin{aligned}v_1 = \ & s_{e_1-1}s_{e_1-2}\cdots s_2s_1\omega s_{n-1}s_{n-2}\cdots s_{e_h}\\ &\times\hat{s}_{e_h-1}s_{e_h-2}\cdots s_{e_{h-1}}\hat{s}_{e_{h-1}-1}s_{e_{h-1}-2}\cdots s_{e_{h-2}}\\ &\times\cdots\hat{s}_{e_2-1}s_{e_2-2}\cdots s_{e_1}\end{aligned}$$

where $\hat{s}_k$ means s_k is omitted. According to 5.5 (a)-(c) and 5.5 (g) we have $v = v_1 w_\lambda$ and $l(v) = l(v_1) + l(w_\lambda)$.

Now we compute $\tilde{T}_u\tilde{T}_{v_1}$. To avoid complicated subscripts we also use $\tilde{T}(w)$ for $\tilde{T}_w$ for any $w \in W$. Set

$$\begin{aligned} v_j = s_{e_j-1}s_{e_j-2}\cdots s_{e_h} \quad & \hat{s}_{e_h-1}s_{e_h-2}\cdots s_{e_{h-1}} \\ \times \quad & \hat{s}_{e_{h-1}-1}s_{e_{h-1}-2}\cdots s_{e_{h-2}} \\ & \cdots \\ \times \quad & \hat{s}_{e_2-1}s_{e_2-2}\cdots s_{e_1}, \end{aligned}$$

for $j = r, r-1, ..., h+1$, note that $r_i = h$;

$$\begin{aligned} v_j = \quad & \hat{s}_{e_j-1}s_{e_j-2}\cdots s_{e_{j-1}} \\ \times \quad & \hat{s}_{e_{j-1}-1}s_{e_{j-1}-2}\cdots s_{e_{j-2}} \\ & \cdots \\ \times \quad & \hat{s}_{e_2-1}s_{e_2-2}\cdots s_{e_1}. \end{aligned}$$

for $j = 2, 3, ..., h$.

For simplicity we often use $i_1 i_2 \cdots i_k$ for $s_{i_1}s_{i_2}\cdots s_{i_k}$. Since $us_q \le u$ for all $1 \le q \le e_1 - 1 = \lambda_1 - 1$, according to Lemma 6.1.3 we have

$$\tilde{T}_u\tilde{T}_v = \sum_{e_1-1\ge j_{1,k_1}>j_{1,k_1-1}>\cdots>j_{1,1}\ge 1} \xi^{\lambda_1-1-k_1}\tilde{T}(uj_{1,k_1}j_{1,k_1-1}\cdots j_{1,1}\omega)\tilde{T}(v_r),$$

where ξ stands for $q - q^{-1}$.

Let

$$u_1 = uj_{1,k_1}j_{1,k_1-1}\cdots j_{1,1}\omega.$$

Recall that $a_{jk} = e_{j-1} + k$ for all $1 \le j \le r$ and $1 \le k \le \lambda_j$. We have

(a) $u_1(n) = u(a_{1l_1}) + n$ for some $1 \le l_1 \le \lambda_1$.

It is easy to see

(b) $l_1 \le k_1 + 1$. Moreover $l_1 = k_1 + 1$ if and only if $j_{1,q} = q$ for $q = 1, ..., k_1$.

(c) $u_1(a) = u(a+1)$ for all $e_1 \le a \le n-1$.

According to Lemma 5.1.1 (c) we have

$$u_1(n) = u(a_{1l_1}) + n > u(a_{ml_1}) = u_1(a_{ml_1} - 1)$$

for $m = 2, 3, ..., r$. For convenience we set

$$u(a_{m0}) = \infty \quad \text{and} \quad u(a_{m,\lambda_m+1}) = -\infty$$

for all m. Choose $0 \le m_r \le \lambda_r$ such that

$$u(a_{rm_r}) > u(a_{1l_1}) + n > u(a_{r,m_r+1}).$$

Then $m_r \le l_1 - 1$. Set

$$u'_r = u_1 s_{e_r-1}s_{e_r-2}\cdots s_{e_{r-1}+m_r}.$$

Then $l(u'_r) = l(u_1) + \lambda_r - m_r$. Moreover we have $u'_r s_q \le u'_r$ for $q = e_{r-1}, e_{r-1}+1, ..., e_{r-1}+m_r-1 = a_{rm_r}-1$. Thus, using Lemma 6.1.3 we get

$$\begin{aligned} \tilde{T}_{u_1}\tilde{T}_{v_r} &= \tilde{T}(u'_r)\tilde{T}(s_{e_{r-1}+m_r-1}\cdots s_{e_{r-1}})\tilde{T}(v_{r-1}) \\ &= \sum \xi^{m_r-k_r}\tilde{T}(u'_r j_{r,k_r}j_{r,k_r-1}\cdots j_{r,1})\tilde{T}(v_{r-1}), \end{aligned}$$

where the sum is for all sequences $e_{r-1}+m_r-1 \ge j_{r,k_r} > j_{r,k_r-1} > \cdots > j_{r,1} \ge e_{r-1}$.

Set
$$u_r = u'_r j_{r,k_r} j_{r,k_r-1}\cdots j_{r,1}.$$
We have

(d) If $m_r = k_r$ then $u_r(e_{r-1}) = u(a_{1l_1}) + n$, in this case we set $l_r = m_r + 1$. If $0 \le k_r < m_r$ then $u_r(e_{r-1}) = u(a_{rl_r})$ for some $1 \le l_r \le k_r + 1$. Note that in any case the $m_r - l_r + 1$ elements
$$u(a_{rm_r}),\ u(a_{r,m_r-1}),\ ...,\ u(a_{r,l_r+1}),\ u_r(e_{r-1})$$
are bigger than $u(a_{1l_1}) + n$, and the $m_r - l_r + 1$ elements
$$u(a_{rm_r}),\ u(a_{r,m_r-1}),\ ...,\ u(a_{r,l_r+1}),\ u_1(e_r)$$
are less than $u_r(e_{r-1})$.

(e) $u_r(a) = u(a+1)$ for all $e_1 \le a \le e_{r-1} - 1$.

Suppose that we have defined $u'_{c'}, u_{c'}, m_{c'} \ge k_{c'} \ge 0, l_{c'}$ for all $r \ge c' > c \ge r_i + 1 = h + 1$ such that

(f) $u_{c'}(e_{c'-1}) = u(a_{c',l_{c'}})$ for some $1 \le l_{c'} \le k_{c'} + 1$ if $k_{c'} < m_{c'}$; and $u_{c'}(e_{c'-1}) = u_{c'+1}(e_{c'})$ if $m_{c'} = k_{c'}$, in this case we set $l_{c'} = m_{c'} + 1$, (we understand that $u_{r+1} = u_1$;)

(g) $u_{c'}(a) = u(a+1)$ for all $e_1 \le a \le e_{c'-1} - 1$.

Now we define u_c, u'_c, k_c, l_c, m_c as follows.

Choose $0 \le m_c \le \lambda_c$ such that
$$u(a_{cm_c}) > u_{c+1}(e_c) > u(a_{c,m_c+1}) = u_{c+1}(a_{c,m_c}).$$
Set
$$u'_c = u_{c+1} s_{e_c-1} s_{e_c-2} \cdots s_{e_{c-1}+m_c}.$$
Then $l(u'_c) = l(u_{c+1}) + \lambda_c - m_c$ and $u'_c s_q \le u'_c$ for $q = e_{c-1}, e_{c-1}+1, ..., e_{c-1}+m_c-1$. Using Lemma 6.1.3 we see
$$\begin{aligned}\tilde T_{u_{c+1}} \tilde T_{v_c} &= \tilde T(u'_c)\tilde T(s_{e_{c-1}+m_c-1}\cdots s_{e_{c-1}})\tilde T(v_{c-1})\\ &= \sum \xi^{m_c-k_c}\tilde T(u'_c j_{c,k_c} j_{c,k_c-1}\cdots j_{c,1})\tilde T(v_{c-1}),\end{aligned}$$
where the sum is for all sequences $e_{c-1}+m_c-1 \ge j_{c,k_c} > j_{c,k_c-1} > \cdots > j_{c,1} \ge e_{c-1}$. Set
$$u_c = u'_c j_{c,k_c} j_{c,k_c-1}\cdots j_{c,1}.$$

If $k_c < m_c$ we can find $1 \le l_c \le k_c + 1$ such that $u_c(e_{c-1}) = u(a_{cl_c})$; if $m_c = k_c$ we set $l_c = m_c + 1$, and in this case we have $u_c(e_{c-1}) = u_{c+1}(e_c)$. Obviously we have

(h) The $m_c - l_c + 1$ elements
$$u(a_{cm_c}),\ u(a_{c,m_c-1}),\ ...,\ u(a_{c,l_c+1}),\ u_c(e_{c-1})$$
are bigger than $u_{c+1}(e_c)$, and the $m_c - l_c + 1$ elements
$$u(a_{cm_c}),\ u(a_{c,m_c-1}),\ ...,\ u(a_{c,l_c+1}),\ u_{c+1}(e_c)$$
are less than $u_c(e_{c-1})$.

(i) $u_c(a) = u(a+1)$ for all $e_1 \le a \le e_{c-1} - 1$.

In this way we defined u_c, u'_c, l_c, k_c, m_c for all $r \geq c \geq r_i + 1 = h + 1$. From (i) we see

(j) $u_{h+1}s_j \leq u_{h+1}$ if $e_1 \leq j \leq e_h - 2$ and $j \neq e_2 - 1, e_3 - 1, ..., e_{h-1} - 1$.

By Lemma 6.1.3 we have

$$\begin{aligned}\tilde{T}_{u_{h+1}}\tilde{T}_{v_h} \quad &= \sum_{\substack{h \geq c \geq 2 \\ e_c - 2 \geq j_{c,k_c} > j_{c,k_c-1} > \cdots > j_{c,1} \geq e_{c-1}}} \xi^{\sum_{h \geq c \geq 2}(\lambda_c - 1 - k_c)} \\ &\quad \times \tilde{T}(u_{h+1} \prod_{h \geq c \geq 2} (j_{c,k_c} j_{c,k_c-1} \cdots j_{c,1})).\end{aligned}$$

Write

$$w' = u_{h+1} \prod_{h \geq c \geq 2} j_{c,k_c} j_{c,k_c-1} \cdots j_{c,1}.$$

We are concerned with the degree of $f_{u,v_1,w'}$. From the construction above we see

(k) The degree of $f_{u,v_1,w'}$ is

$$\lambda_1 - 1 - k_1 + \sum_{r \geq c \geq h+1} (m_r - k_r) + \sum_{h \geq c \geq 2} (\lambda_c - 1 - k_c).$$

We have

$$\begin{aligned}\tilde{T}_u\tilde{T}_v &= \tilde{T}_u\tilde{T}_{v_1}\tilde{T}_{w_\lambda} \\ &= \textstyle\sum_{w' \in W} f_{u,v_1,w'}\tilde{T}_{w'}\tilde{T}_{w_\lambda} \\ &= \textstyle\sum_{w',w \in W} f_{u,v_1,w'} f_{w',w_\lambda,w}\tilde{T}_w.\end{aligned}$$

Now we consider the degree of $f_{w',w_\lambda,w}$. Note that $w'(e_c) = u(a_{c+1,l_{c+1}})$ for some $1 \leq l_{c+1} \leq \lambda_{c+1}$ whenever $1 \leq c \leq h - 1$. Obviously we have

(l) $l_{c+1} \leq k_{c+1} + 1$ for $1 \leq c \leq h - 1$.

According to Lemma 5.4.4 we have

(m) $u(a_{c+1,l_{c+1}}) > u(a_{c,l_{c+1}})$ if $1 \leq c \leq h - 1$.

Note that if $1 \leq c \leq h-1$ then $w'(a_{c1}), ..., w'(a_{c,\lambda_c-1})$ are in $\{u(a_{c1}), ..., u(a_{c\lambda_c})\}$. Thus,

(n) among $w'(a_{c1}), ..., w'(a_{c,\lambda_c-1})$ at least $\lambda_c - (l_{c+1} - 1) - 1 = \lambda_c - l_{c+1}$ of them are less than $w'(e_c) = u(a_{c+1,l_{c+1}})$.

(o) Suppose that among $u(a_{h1}), ..., u(a_{h\lambda_h})$ we have that β of them are less than $w'(a_{h\lambda_h}) = u_{h+1}(e_h)$. Since $w'(a_{h1}), ..., w'(a_{h,\lambda_h-1})$ are in $\{u(a_{h1}), ..., u(a_{h\lambda_h})\}$ we see at least $\beta - 1$ of $w'(a_{h1}), ..., w'(a_{h,\lambda_h-1})$ are less than $w'(a_{h\lambda_h})$.

Using (h) and (o) we get an r-chain of u consisting of

$$u(a_{h1}),\ ...,\ u(a_{h,\lambda_h-\beta}),\ u_{h+1}(e_h),\ u(a_{h+1,l_{h+1}+1}),\ ...,\ u(a_{h+1,m_{h+1}}),$$

$$...,\ u_r(e_{r-1}),\ u(a_{r,l_r+1}),\ ...,\ u(a_{rm_r}),\ u(a_{1l_1}+n),\ ...,\ u(a_{1\lambda_1}+n),$$

whose length is

$$L = \lambda_h - \beta + m_{h+1} - l_{h+1} + 1 + \cdots + m_r - l_r + 1 + \lambda_1 - l_1 + 1.$$

If

$$-(\beta-1)+m_r-k_r+\cdots+m_{h+1}-k_{h+1}+\lambda_h-1-k_1>0,$$

then we have

$$L\geq \lambda_h-\beta+m_{h+1}-k_{h+1}+\cdots+m_r-k_r+\lambda_1-(k_1+1)+1>\lambda_1.$$

This is impossible since the partition of u is λ. So we have

(p) $-(\beta-1)+m_r-k_r+\cdots+m_{h+1}-k_{h+1}+\lambda_h-1-k_1\leq 0$

Write $w'=w_1'w_2'$ such that w_1' is the shortest element in $w'W_\lambda$ and w_2' is in W_λ. According to (n) and (o), we have

(q) $l(w_2')\leq l(w_\lambda)-\sum_{1\leq c\leq h-1}(\lambda_c-l_{c+1})-(\beta-1)$.

Using Lemma 6.1.4 we see that the degree of $f_{w',w_\lambda,w}$ is less than or equal to $l(w_\lambda)-\sum_{1\leq c\leq h-1}(\lambda_c-l_{c+1})-(\beta-1)$.

(r) Since the degree of $f_{u,v,w}$ is the maximal number in

$$\{\deg f_{u,v_1,w'}+\deg f_{w',w_\lambda,w}\mid w'\in W\},$$

we see that the degree of $f_{u,v,w}$ is less than or equal to

$$\begin{array}{rl} & \lambda_1-k_1-1+m_r-k_r+\cdots+m_{h+1}-k_{h+1}\\ + & \lambda_h-1-k_h+\cdots+\lambda_2-1-k_2\\ + & l(w_\lambda)-(\lambda_{h-1}-l_h)-\cdots-(\lambda_1-l_2)-(\beta-1)\\ \leq & l(w_\lambda)=a(w_\lambda).\end{array}$$

The Proposition is proved.

Corollary 6.2.3. *Keep the notation in 6.2.2. Suppose that $f_{u,v,w}$ has degree $a(w_\lambda)$. Then we have*

(a) *l_c is equal to k_c+1 for $1\leq c\leq h$.*
(b) *$u(a_{c-1,l_c-1})>u(a_{cl_c})>u(a_{c-1,l_c})$ for $2\leq c\leq h$. (Note that $h=r_i$.)*
(c) *$l_c\leq l_{c-1}$ for $2\leq c\leq h$.*
(d) *$-(\beta-1)+m_r-k_r+\cdots+m_{h+1}-k_{h+1}+\lambda_h-1-k_1=0$.*
(e) *$u_{h+1}(e_h)>u(a_{hl_h})$ and $u(a_{cl_c})>u(a_{c-1,l_{c-1}})$ for $c=2,3,...,h$.*

Proof. (a) If $l_c<k_c+1$ for some c, then $w'(a)<w'(b)$ for some

$$e_{c-1}+l_c+1\leq a<b\leq e_{c-1}+k_c+1<e_c.$$

Thus $l(w_2')$ would be less than

$$l(w_\lambda)-\sum_{1\leq c\leq h-1}(\lambda_c-l_{c+1})-(\beta-1).$$

From the proof in 6.2.2 we see that the degree of $f_{u,v,w}$ then would be less than $a(w_\lambda)$. (When $2\leq c\leq h$ we can see that $l_c=k_c+1$ using 6.2.2 (l) and 6.2.2 (r).)

(b) By Lemma 5.4.4 we have $u(a_{cl_c})>u(a_{c-1,l_c})$. Suppose that $u(a_{cl_c})>u(a_{c-1,l_c-1})$ for some c with $2\leq c\leq h$. Then among $w'(a_{c-1,1}),...,w'(a_{c-1,\lambda_{c-1}-1})$ at least $\lambda_{c-1}-(l_c-2)-1=\lambda_{c-1}-l_c+1$ of them are less than $w'(a_{c-1,l_{c-1}})$. Then $l(w_2')$ would be less than $l(w_\lambda)-\sum_{1\leq c\leq h-1}(\lambda_c-l_{c+1})-(\beta-1)$. From 6.2.2 (l) and 6.2.2 (p-r) we see that the degree of $f_{u,v,w}$ then would be less than $a(w_\lambda)$. Therefore (b) is true.

(c) The proof is similar. If $l_c > l_{c-1}$ for some $2 \leq c \leq h$, then among $w'(a_{c-1,1})$,...,$w'(a_{c-1,\lambda_{c-1}-1})$ at least $\lambda_{c-1} - (l_c - 2) - 1 = \lambda_{c-1} - l_c + 1$ of them are less than $w'(a_{c-1,l_{c-1}})$. As the proof of (b), this is impossible. Therefore (c) is true.

(d) It is clear from the proof of Prop. 6.2.1.

(e) When $c = 2, 3, ..., h$, using (c) and Lemma 5.4.4, we see that $u(a_{cl_c}) > u(a_{c-1,l_{c-1}})$. If $u_{h+1}(e_h) < u(a_{hl_h})$, then β of $w'(a_{h1}), ..., w'(a_{h,\lambda_h-1})$ are less than $w'(a_{h\lambda_h})$. Thus $l(w_2')$ would be less than $l(w_\lambda) - \sum_{1\leq c\leq h-1}(\lambda_c - l_{c+1}) - (\beta - 1)$. By 6.2.2 (q-r), this contradicts that $f_{u,v,w}$ has degree $a(w_\lambda)$. Therefore $u_{h+1}(e_h) > u(a_{hl_h})$. (e) is proved.

The proof is completed.

Corollary 6.2.4. *Keep the notation in 6.2.2. Suppose that* $degf_{u,v,w} = a(w_\lambda)$.

(a) *We have* $w(a_{cl}) > w(a_{cl'})$ *for any* $1 \leq c \leq r$ *and* $1 \leq l < l' \leq \lambda_c$.

(b) *Suppose that* $1 \leq c < h$. *Then*

$$w(a_{cl}) = \begin{cases} u(a_{cl}) & \text{if } 1 \leq l \leq l_{c+1} - 1, \\ u(a_{c+1,l_{c+1}}) & \text{if } l = l_{c+1}, \\ u(a_{c,l-1}) & \text{if } l_{c+1} < l \leq l_c \\ u(a_{cl}) & \text{if } l_c < l \leq \lambda_c. \end{cases}$$

(c)

$$w(a_{hl}) = \begin{cases} u(a_{hl}) & \text{if } 1 \leq l \leq \lambda_h - \beta, \\ w'(e_h) & \text{if } l = \lambda_h - \beta + 1, \\ u(a_{h,l-1}) & \text{if } \lambda_h - \beta + 1 < l \leq l_h, \\ u(a_{hl}) & \text{if } l_h < l \leq \lambda_h. \end{cases}$$

(d) *Suppose that* $h + 1 \leq c \leq r$. *We have*

$$w(a_{cl}) = \begin{cases} u(a_{cl}) & \text{if } 1 \leq l < l_c, \\ u(a_{c,l+1}) & \text{if } l_c \leq l < m_c, \\ u_{c+1}(e_c) & \text{if } l = m_c, \\ u(a_{cl}) & \text{if } m_c < l \leq \lambda_c. \end{cases}$$

Proof. (a) From the proof of Prop. 6.2.1 we see $l(w) = l(ww_\lambda) + l(w_\lambda)$. Using Lemma 2.5.1 we get (a).

(b) Using Corollary 6.2.3 (b)-(c) we see that (b) is true.

(c) and (d) are clear from the proof of Prop. 6.2.1.

The corollary is proved.

6.3. Some consequences

Let u, v be as in section 6.1. In this section we will figure out the w in $\Gamma_\lambda \cap \Gamma_\lambda^{-1}$ with $f_{u,v,w}$ having degree $a(w_\lambda)$. We keep the notation in 6.2.2.

Lemma 6.3.1. *If $\beta = 0$ then $\deg f_{u,v,w}$ is less than $a(w_\lambda)$.*

Proof. In 6.2.2 we have showed that

$$-(\beta-1) + m_r - k_r + \cdots + m_{h+1} - k_{h+1} + \lambda_h - 1 - k_1 \leq 0.$$

When $\beta = 0$, we have

$$m_r - k_r + \cdots + m_{h+1} - k_{h+1} + \lambda_h - k_1 \leq 0.$$

Therefore the degree of $f_{u,v,w}$ is either less than or equal to

$$\begin{aligned} & \lambda_1 - k_1 - 1 + m_r - k_r + \cdots + m_{h+1} - k_{h+1} \\ & + \lambda_h - 1 - k_h + \cdots + \lambda_2 - 1 - k_2 \\ & + l(w_\lambda) - (\lambda_{h-1} - l_h) - \cdots - (\lambda_1 - l_2) \\ < \quad & l(w_\lambda) = a(w_\lambda). \end{aligned}$$

The lemma is proved.

Lemma 6.3.2. *Assume that $\deg f_{u,v,w} = a(w_\lambda)$.*
(a) *If $l_1 > \lambda_h$ then w is not in $\Gamma_\lambda \cap \Gamma_\lambda^{-1}$.*
(b) If w is in $\Gamma_\lambda \cap \Gamma_\lambda^{-1}$, then $\varepsilon_k(u)$ is equal to $\varepsilon_k(w)$ if $1 \leq k \leq i-1$.

Proof. (a) According to Corollary 6.2.3 (d) we must have

$$-(\beta-1) + m_r - k_r + \cdots + m_{h+1} - k_{h+1} + \lambda_h - 1 - k_1 = 0.$$

Thus the r-chain of w, (here we need Corollary 6.2.4 and 6.2.2 (h))

$$\begin{aligned} & w(a_{h1}), \quad ..., \quad w(a_{h,\lambda_h-\beta}), \quad w(a_{h,\lambda_h-\beta+1}), \\ & w(a_{h+1,l_{h+1}}), \quad ..., \quad w(a_{h+1,m_{h+1}}), \\ & \qquad ..., \\ & w(a_{r,l_r}), \quad ..., \quad w(a_{r,m_r}), \end{aligned}$$

has length

$$L = \lambda_h - \beta + 1 + m_{h+1} - l_{h+1} + 1 + \cdots + m_r - l_r + 1.$$

We have

$$\begin{aligned} L \quad & \geq \lambda_h - \beta + 1 + m_{h+1} - k_{h+1} + \cdots + m_r - k_r \\ & = k_1 + 1 \\ & \geq l_1 \\ & > \lambda_h. \end{aligned}$$

By Corollary 6.2.4 (a), the sequence $w(a_{c1}) > \cdots > w(a_{c\lambda_c})$ is an r-chain of w of length λ_c if $1 \leq c \leq h-1$. Thus w has an r-chain family set of index h and the cardinality of the r-chain family set is bigger than e_h. Therefore $w \notin \Gamma_\lambda \cap \Gamma_\lambda^{-1}$.

(b) Since $\varepsilon_k(u) = 0$ for $k = 1, ..., i-1$ and all components of $\varepsilon(u)$ are non-negative, using Lemma 5.3.2, Theorem 5.1.12 and Lemma 5.1.4 we see that

$$1 \le u(a_{pq}) \le n \quad \text{if} \quad 1 \le p \le r_{i-1} \text{ and } q > \lambda_h. \tag{1}$$

By (a) and Corollary 6.2.3 (c) we get

$$\lambda_h \ge l_1 \ge l_2 \ge \cdots \ge l_h. \tag{2}$$

Using Corollary 6.2.4 we then get

$$w(a_{pq}) = u(a_{pq}) \quad \text{if} \quad 1 \le p \le r_{i-1} \text{ and } q > \lambda_h. \tag{3}$$

Using Lemma 5.4.4 for u and Corollary 6.2.4 we see clearly that

$$w(a_{pq}) > w(a_{p-1,q}) \quad \text{if} \quad 2 \le p \le h-1 \text{ and } 1 \le q \le \lambda_p, \lambda_{p-1}. \tag{4}$$

Now we show that

$$w(a_{hq}) > w(a_{h-1,q}) \quad \text{if} \quad 1 \le q \le \lambda_h, \lambda_{h-1}. \tag{5}$$

We must have $l_h \ge \lambda_h - \beta + 1$ since $\deg f_{u,v,w} = a(w_\lambda)$. Using Lemma 5.4.4 and Corollary 6.2.4 we see that (5) is true.

Using (1), (3-5), Lemma 5.1.4 and Lemma 5.2.3, we see that

$$\varepsilon_k(w) = (0, ..., 0) = \varepsilon_k(u)$$

if $1 \le k \le i-1$. (b) is proved.

The lemma is proved.

Definition 6.3.3. Let $a \in \mathbb{Z}$. Write $a = a_{ij} + kn,\ 1 \le i \le r,\ 1 \le j \le \lambda_i$. The **level** of a is defined to be the pair (i, k). We say that $(i, k) > (i', k')$ if $k > k'$ or $i > i'$ and $k = k'$.

Let $w \in W$. An r-antichain of w is called **saturated** if the r-antichain is contained in some complete r-antichain family of w. Analogously we define **saturated d-antichains** of w.

Lemma 6.3.4. *If w is in $\Gamma_\lambda \cap \Gamma_\lambda^{-1}$ and $\deg f_{u,v,w} = a(w_\lambda)$, then all*

$$u(a_{cl_c}),\ u(a_{c,l_c+1}),\ ...,\ u(a_{cm_c}),\ u_{c+1}(e_c)$$

have the same level whenever $h+1 \le c \le r$ and $l_c \le m_c$. (Recall that $u_{r+1} = u_1$.)

Proof. Suppose that the conclusion is not true. Then there exists some c with $h+1 \le c \le r$ and $l_c \le m_c$ such that some pair of neighboring terms in the sequence

$$u(a_{cl_c}),\ u(a_{c,l_c+1}),\ ...,\ u(a_{cm_c}),\ u_{c+1}(e_c)$$

have distinct levels but all

$$u(a_{c'l_{c'}}),\ u(a_{c',l'_c+1}),\ ...,\ u(a_{c'm_{c'}}),\ u_{c'+1}(e_{c'})$$

have the same level for all $c < c' \le r$ with $l_{c'} \le m_{c'}$.

Let Z be a complete r-antichain family of u. For $\theta = l_c, l_{c+1}, ..., m_c$ we choose β_θ such that $u(a_{c-1,\beta_\theta})$ and $u(a_{c\theta})$ are in an r-antichain of u that is contained in Z.

Suppose that all

$$u(a_{cl_c}), u(a_{c,l_c+1}), ..., u(a_{c,\theta-1})$$

have the same level but $u(a_{c,\theta-1})$ and $u(a_{c\theta})$ have distinct levels for some $l_c < \theta \le m_c$. Obviously we have

(a) $u(a_{c,\theta-1}) > u(a_{c\theta})$.

Suppose that $c-1>h$. We claim that

(b) $u(a_{c-1,\beta_{\theta-1}}) > u(a_{c\theta})$.

Write

$$u(a_{cl_c}) = \xi+\eta n,\quad 1\le\xi\le n,\ \eta\in\mathbb{Z}.$$

Then $\xi\in\Lambda_q$ for some $1\le q\le r$. By assumption, $w(a_{kl})$ and $u(a_{kl})$ have the same level for all $c<k\le r$ and $1\le l\le\lambda_k$. By Lemma 5.1.4, then we may assume that the r-antichain C in Z containing $u(a_{cl_c})$ has length c and there is a saturated r-antichain of w that contains $u_{c+1}(e_c)$ and has length c.

Note that $u(a_{c-1,\beta_{l_c}})$ is in C and $u_{c-1}(e_{c-2})\ge u(a_{cl_c})$ (since $c-1>h$). Using Lemma 5.1.4, then we can find a saturated r-antichain of w that contains $u(a_{c-1,\beta_{l_c}})$ and has length $c-1$ since $u(a_{cl_c})$ and $u_{c+1}(e_c)$ have distinct levels.

We claim that $q>1$. Otherwise, $q=1$. Write

$$u(a_{c-1,\beta_{l_c}}) = x+yn,\quad 1\le x\le n,\quad y\in\mathbb{Z}.$$

By Lemma 5.1.4, we have $x\in\Lambda_c$. But there is a saturated r-antichain of w that contains $u(a_{c-1,\beta_{l_c}})$ and has length $c-1$, by Lemma 5.1.4, this is impossible. Therefore $q>1$ and $x\in\Lambda_{q-1}$ and $y=\eta$. Using Lemma 5.1.4 we see that $u(a_{c-1,\beta_{l_c}})$ and $u(a_{c-1,\beta_{\theta-1}})$ have the same level. Write

$$u(a_{c,\theta}) = \xi'+\eta' n,\quad 1\le\xi'\le n,\quad \eta'\in\mathbb{Z},$$

and

$$u(a_{c-1,\beta_{\theta-1}}) = x'+y'n,\quad 1\le x'\le n,\quad y'\in\mathbb{Z}.$$

then $x'\in\Lambda_{q-1}$ and $y=y'$.

Since $u(a_{c,\theta-1})>u(a_{c\theta})$, $y'=\eta$, and $u(a_{cl_c})$, $u(a_{c,\theta-1})$ have the same level, we have $y'\ge\eta'$ and $\xi'\in\Lambda_{q'}$ for some $1\le q'<q$ if $y'=\eta'$.

When $y'>\eta'$, clearly (b) is true. If $y'=\eta'$ and $1\le q'<q-1$, (b) is also obviously true. Now suppose that $y'=\eta'$ and $q'=q-1$. Using Lemma 5.4.2 we see that (b) is true. Thus (b) is always true.

Similarly if $u(a_{cl_c}),...,u(a_{cm_c})$ have the same level and $u(a_{cm_c}),u_{c+1}(e_c)$ have distinct levels and $u_{c+1}(e_c)=u(a_{kl})$ for some $c<k\le r$ and $1\le l\le\lambda_k$, we have $u(a_{c-1,\beta_{m_c}})>u_{c+1}(e_c)$. If $u(a_{cl_c}),...,u(a_{cm_c})$ have the same level but $u(a_{cm_c})$ and $u_{c+1}(e_c)$ have distinct levels and $u_{c+1}(e_c)=u(a_{1l_1}+n)$, using $u^{-1}\in\Gamma_\lambda\cap\Gamma_\lambda^{-1}$ we can see that $u(a_{c-1,\beta_{m_c}})>u_{c+1}(e_c)$. Thus in this case (b) is true.

Note that $u(a_{cl_c})>u(a_{c-1,\beta_\theta})$ for all β_θ. Thus $m_{c-1}<\beta_\theta$ for all β_θ since $c-1\ge h+1$. Using Corollary 6.2.4 we see that the following elements

$$w(a_{h1}),\ ...,\ w(a_{h,\lambda_h-\beta}),\ w(a_{h,\lambda_h-\beta+1}),$$

$$w(a_{h+1,l_{h+1}}),\ ...,\ w(a_{h+1,m_{h+1}}),\ \ ...,\ \ w(a_{c-2,l_{c-2}}),\ ...,\ w(a_{c-2,m_{c-2}}),$$

$$w(a_{c-1,l_{c-1}}),\ ...,\ w(a_{c-1,m_{c-1}}),\ w(a_{c-1,\beta_{l_c}}),\ w(a_{c-1,\beta_{l_c+1}}),\ ...,\ w(a_{c-1,\beta_{\theta-1}}),$$

$$w(a_{c,\theta-1}),\ ...,\ w(a_{cm_c}),$$

$$w(a_{c+1,l_{c+1}}),\ ...,\ w(a_{c+1,m_{c+1}}),\ \ ...,\ \ w(a_{r-1,l_{r-1}},\ ...,\ w(a_{r-1,m_{r-1}}),$$

$$w(a_{rl_r}),\ w(a_{r,l_r+1}),\ ...,\ w(a_{r,m_r}),\ w(a_{1,l_1+1}+n),\ ...,\ w(a_{1\lambda_1}+n),$$

form an r-chain of w.

Since

$$-(\beta-1)+m_r-k_r+\cdots+m_{h+1}-k_{h+1}+\lambda_h-1-k_1=0,$$

the above r-chain of w has length

$$\begin{aligned} L = \quad & \lambda_h-\beta+1+m_{h+1}-l_{h+1}+1+\cdots+m_{c-1}-l_{c-1}+1 \\ & +m_c-l_c+2+m_{c+1}-l_{c+1}+1+\cdots+m_r-l_r+1+\lambda_1-l_1. \end{aligned}$$

We have

$$\begin{aligned} L \quad & \geq \lambda_h-\beta+1+m_{h+1}-k_{h+1}+\cdots+m_{c-1}-k_{c-1} \\ & \quad +m_c-k_c+1+m_{c+1}-k_{c+1}+\cdots+m_r-k_r+1+\lambda_1-k_1-1 \\ & = \lambda_1+1 \\ & > \lambda_1. \end{aligned}$$

This is impossible since $w \in \Gamma_\lambda \cap \Gamma_\lambda^{-1}$, so the lemma is true in this case.

Suppose that $c = h+1$. By assumption, we have

(c) $u(a_{c'l_{c'}})$ and $u_{c'+1}(e_{c'})$ have the same level for all $c = h+1 < c' \leq r$ with $l_{c'} \leq m_{c'}$.

Using Corollary 6.2.4 we get (note that $l_{h+1} \leq m_{h+1}$ in our case)

(d) $u(a_{h+1,l_{h+1}}) = w'(e_h) = w(a_{h,\lambda_h-\beta+1})$, and $w(a_{kl})$ and $u(a_{kl})$ have the same level for all $r \geq k \geq h+2$ and $\lambda_k \geq l \geq 1$. And

$$w(\Lambda_{h+1}) - \{w(a_{h+1,m_{h+1}})\} = u(\Lambda_{h+1}) - \{u(a_{h+1,l_{h+1}})\}.$$

Using Lemma 5.1.4, thus we have

(e) there is a complete r-antichain family Y_u of u such that the r-antichain B in Y_u containing $u(a_{h+1,l_{h+1}})$ has length $h+1$;

(f) there is a complete r-antichain family Y_w of w such that the r-antichain C in Y_w containing $w(a_{h,\lambda_h-\beta+1}) = u(a_{h+1,l_{h+1}})$ has length h; and the r-antichain D in Y_w containing $w(a_{h+1,m_{h+1}}) = u_{h+2}(e_{h+1})$ has length $h+1$.

By Corollary 6.2.3 (e), $u(a_{kl_k}) < u(a_{k+1,l_{k+1}})$ for $k = 1,2,...,h$. Using Lemma 5.4.2 we see

(g) $u(a_{jl_j})$ and $u(a_{kl_k})$ have distinct levels whenever $1 \leq j < k \leq h+1$.

Let B' be an r-antichain in Y_u. Suppose that B' has length greater than $h+1$ or of length $h+1$ but does not contain $u(a_{h+1,l_{h+1}})$. Assume that $u(a_{kl_k})$ is in B' for some $1 \leq k \leq h$. By (g) and (d), we can find some $u(a_{k\theta_k})$ that has the same level as $u(a_{kl_k})$ and in an r-antichain B'' in Y_u of length less than the length of B'. Replacing $u(a_{kl_k})$ by $u(a_{k\theta_k})$ in B' and replacing $u(a_{k\theta_k})$ by $u(a_{kl_k})$ in B'', and continuing this process, finally we get a complete r-antichain family Y'_u of u with the following property.

(h) Any r-antichain in Y'_u of length greater than $h+1$ or of length $h+1$ but not containing $u(a_{h+1,l_{h+1}})$, does not contain any $u(a_{kl_k})$ for $k = 1,2,...,h$. Moreover the r-antichain in Y'_u containing $u(a_{h+1,l_{h+1}})$ has length $h+1$.

(i) It is harmless to require that $Y_u = Y'_u$. Then, for any r-antichain B' in Y_u that has length greater than $h+1$ or has length $h+1$ but does not contain $u(a_{h+1,l_{h+1}})$, replacing $u(a_{c'l_{c'}})$ $(h+1 < c' \leq r)$ by $w(a_{c'm_{c'}})$ if $u(a_{c'l_{c'}})$ is B', we get an r-antichain C' of w. It is harmless to assume all such C' are in Y_w.

Suppose that $u(a_{kq_k})$ are in B for $k=1,2,...,h$. Write

$$u(a_{h+1,l_{h+1}})=x_{h+1}+y_{h+1}n,\quad 1\le x_{h+1}\le n,\quad y_{h+1}\in\mathbb{Z},$$
$$u(a_{kq_k})=x_k+y_kn,\quad 1\le x_k\le n,\quad y_k\in\mathbb{Z},\quad k=1,2,...,h.$$

By (e), (f) and Lemma 5.1.4 we get

(j) x_{h+1} is in Λ_g for some $1\le g\le h$ and x_{h+1-g} is in Λ_{h+1}.

Since

$$u(a_{h+1,l_{h+1}})>u_{h+2}(e_{h+1})=w(a_{h+1,m_{h+1}})$$

and they have distinct levels, by Corollary 6.2.4, if $q_{h+1-g}\ne l_{h+1-g}$ then $u(a_{h+1-g,q_{h+1-g}})$ is in $w(\Lambda_{h+1-g})$, if $h+1-g\ge 2$ then $u(a_{h+1-g,q_{h+1-g}})$ is in $w(\Lambda_{h+1-g})$ or in $w(\Lambda_{h+1-g-1})$. Then in both cases, by (i) and (f), $u(a_{h+1-g,q_{h+1-g}})$ is not in any r-antichain in Y_w of length greater than h. By Lemma 5.1.4, this is impossible. Hence $h+1-g=1$ and $q_1=l_1$. But we have $u(a_{h+1,l_{h+1}})>u_{h+2}(e_{h+1})\ge u(a_{1l_1})+n$, this contradicts that B is an r-antichain of u. Therefore $u(a_{h+1,l_{h+1}})$ and $u_{h+2}(e_{h+1})=w(a_{h+1,m_{h+1}})$ have the same level.

The lemma is proved

Corollary 6.3.5. *$\varepsilon_k(u)=\varepsilon_k(w)$ whenever $k\ne i$.*

Proof. When $k<i$, this is Lemma 6.3.2 (b). Using Lemma 6.3.4 and Corollary 6.2.4 we see that $w(a_{cl})$ and $u(a_{cl})$ have the same level if $c>r_i$ and $\lambda_c\ge l\ge 1$. Therefore $\varepsilon_k(u)=\varepsilon_k(w)$ if $k>i$. The corollary is proved.

Lemma 6.3.6. *Let $u,v\in\Gamma_\lambda\cap\Gamma_\lambda^{-1}$ be as in Prop. 6.2.1. Keep the notation in 6.2.2. Assume that $w\in\Gamma_\lambda\cap\Gamma_\lambda^{-1}$ and $\deg f_{u,v,w}=a(w_\lambda)$. Then*
(a) $u(a_{1l_1})$ and $u(a_{1,l_1-1})$ have distinct levels and $1\le l_1\le\lambda_h$.
(b) $u(a_{cl_c})$ and $u(a_{c,l_c-1})$ have distinct levels for $c=2,3,...,r$.
(c) There is a complete r-antichain family Z of u such that the r-antichain B in Z containing $u(a_{1l_1})$ has length h. (Recall that $r_i=h$.)

Proof. (a) By Lemma 6.3.2 (a) we have $1\le l_1\le\lambda_h$. If $u(a_{1,l_1-1})$ and $u(a_{1l_1})$ have the same level then

$$u(a_{1,l_1-1})=a_{kl}+qn\quad\text{ and }u(a_{1l_1})=a_{kl'}+qn$$

for some $r\ge k\ge 1$, $\lambda_k\ge l>l'\ge 1$ and some $q\in\mathbb{Z}$. Then $w^{-1}(a_{kl'})=\xi'-(q+1)n$ for some $1\le\xi'\le n$, and $w^{-1}(a_{kl})=\xi-qn$ for some $1\le\xi\le\lambda_1$. Thus $w^{-1}(a_{kl'})<w^{-1}(a_{kl})$. This contradicts that $w\in\Gamma_\lambda\cap\Gamma_\lambda^{-1}$. Therefore $u(a_{1,l_1-1})$ and $u(a_{1l_1})$ have distinct levels.

(b) Similarly we see that (b) is true.

(c) Let A_u (resp. A_w) be the set of elements in $u(\Lambda_1)$ (resp. $w(\Lambda_1)$) that have the same level as $u(a_{1l_1})$. Let Z be a complete r-antichain family of u. Assume that (c) is not true. By Lemma 5.1.4, then any r-antichain in Z that contains one element in A_u has length different from h. By Corollary 6.2.3 (e) and Lemma 5.4.2, $u(a_{2l_2})$ and $u(a_{1l_1})$ have distinct levels. Now $w(\Lambda_1)$ is the union of $\{u(L_1)-\{u(a_{1l_1})\}$ and $u(a_{2l_2})$. Thus $|A_u|=|A_w|+1$. By Lemma 5.1.4, this forces that $\varepsilon_k(w)\ne\varepsilon_k(u)$ for some $k\ne i$. This is impossible by Corollary 6.3.5. So there must be some r-antichain in a complete r-antichain family of u that hash length h and contains $u(a_{1l_1})$.

Proposition 6.3.7. *Let $u, v \in \Gamma_\lambda \cap \Gamma_\lambda^{-1}$ be such that all components of $\varepsilon(u)$ and $\varepsilon(v)$ are nonnegative. Assume that*

$$\varepsilon_1(u) = \varepsilon_2(u) = \cdots = \varepsilon_{i-1}(u) = 0$$

and

$$\varepsilon_{i1}(v) = 1 \quad \text{and} \quad \varepsilon_{kl}(v) = 0 \quad \text{if } (k,l) \neq (i,1)$$

Then

$$t_u t_v = \sum t_w,$$

where w runs through the set

$$\mathcal{I} = \{w \in \Gamma_\lambda \cap \Gamma_\lambda^{-1} \mid \quad \varepsilon_{ij}(w) = \varepsilon_{ij}(u) + 1 \text{ for some } 1 \leq j \leq n_i, \\ \text{and } \varepsilon_{kl}(w) = \varepsilon_{kl}(u) \text{ if } (k,l) \neq (i,j)\}.$$

Proof. Suppose that $w \in \mathcal{I}$ and $\varepsilon_{ij}(w) = \varepsilon_{ij}(u) + 1$. Then $\varepsilon_{i,j-1}(u) > \varepsilon_{ij}(u)$. (We understand that $\varepsilon_{i0}(u) = \infty$.) Let Z be a complete r-antichain family of u and B an r-antichain in Z that has length r_i and provides $\varepsilon_{ij}(u)$. Let $u(a_{kl_k})$ be in B for $k = 1, ..., h$. We may choose l_k as big as possible. By Lemma 5.1.4, then $u(a_{k,l_k-1})$ and $u(a_{kl_k})$ have distinct levels. According to Lemmas 5.4.4 and 5.2.3, we see that $1 \leq l_1 \leq \lambda_h$.

Now suppose that $1 \leq l_1 \leq \lambda_h$ satisfies (a) and (c) in Lemma 6.3.6. Assume that B is an r-antichain in a complete r-antichain family Z that has length r_i and provides $\varepsilon_{ij}(u)$. Then obviously we have $\varepsilon_{i,j-1}(u) > \varepsilon_{ij}(u)$. We need show that there exist unique $k_1,\ k_c,\ 1 \leq l_c \leq \lambda_c + 1$ for $c = 2, 3, ..., r$ such that $k_c + 1 = l_c$ for all $1 \leq c \leq r$ and the corresponding w in 6.2.2 satisfies that (1) $w \in \Gamma_\lambda \cap \Gamma_\lambda^{-1}$ and (2) $\deg f_{u,v,w} = a(w_\lambda)$.

Keep the notation in 6.2.2.

If $c \geq h+1$, we choose $1 \leq l_c \leq m_c + 1$ inductively so that (1) $u(a_{cm_c}) > u_{c+1}(e_c) > u(a_{c,m_c+1})$, (2) $u(a_{cl_c})$ and $u_{c+1}(e_c)$ have the same level but $u(a_{c,l_c-1})$ and $u(a_{cl_c})$ have distinct levels, noting that if no such l_c we set $l_c = m_c + 1$. Then set $k_c = l_c - 1$.

Let $u(a_{h\alpha_h}), ..., u(a_{2\alpha_2}), u(a_{1l_1})$ be the r-antichain B in Lemma 6.3.6 (c). We may require that $u(a_{c\alpha_c})$ and $u(a_{c,\alpha_c-1})$ have distinct levels for $c = 2, ..., h$. Then we set $l_c = \alpha_c$ for $c = 2, ..., h$.

According to the arguments in 6.2.2, it suffices to prove the following three assertions.

(a) $l_h \leq l_{h-1} \leq \cdots \leq l_1$.

(b) $u(a_{c-1,l_c-1}) > u(a_{cl_c})$ for $c = 2, 3, ..., h$.

(c) Let $1 \leq \beta \leq \lambda_h$ be such that $u(a_{h,\lambda_h-\beta}) > u_{h+1}(e_h) > u(a_{h,\lambda_h-\beta+1})$. Then

$$\lambda_h - \beta + m_{h+1} - l_{h+1} + 1 + \cdots + m_r - l_r + 1 = l_1 - 1.$$

(Note that $u(a_{hl_h}) < u(a_{1l_1}) + n \leq u_{h+1}(e_h)$. So $\lambda_h - \beta + 1 \leq l_h$.)

For all $1 \leq c \leq h$ we write

$$u(a_{cl_c}) = \xi_c + \eta_c n, \qquad \text{where } 1 \leq \xi_c \leq n \text{ and } \eta_c \in \mathbb{Z}.$$

Let Z be a complete r-antichain family of u containing B.

Now we argue for (a). If $l_c > l_{c-1}$ for some $2 \le c \le h$, then there is an r-antichain C in Z such that C contains $u(a_{cp_c})$ and $u(a_{c-1,p_{c-1}})$ for some $1 \le p_c < l_c$ and $l_{c-1} < p_{c-1} \le \lambda_{c-1}$. Write

$$u(a_{cp_c}) = x + yn \quad\text{and}\quad u(a_{c-1,p_{c-1}}) = x' + y'n,$$

where $1 \le x, x' \le n$ and $y, y' \in \mathbb{Z}$. Assume that

$$x \in \Lambda_j, \quad x' \in \Lambda_k \quad\text{and}\quad \xi_c \in \Lambda_q.$$

If $q > 1$, by Lemma 5.1.4, then $\xi_{c-1} \in \Lambda_{q-1}$ and $\eta_c = \eta_{c-1}$. Since $u(a_{cp_c}) \ge u(a_{c,l_c-1})$, $u(a_{c,l_c-1})$ and $u(a_{cl_c})$ have distinct levels, and $u(a_{c,l_{c-1}}) > u(a_{c-1,p_{c-1}})$ (see Lemma 5.4.4), we see

(d) either $j > q$ and $y \ge \eta_c$, or $y > \eta_c$; and either $k \le q-1$ and $y' \le \eta_c$, or $y' < \eta_c$.

If $j > 1$, using Lemma 5.1.4 we see that $k = j-1$ and $y = y'$. This contradicts (d). If $j = 1$, then $y > \eta_c$. By Lemma 5.3.2, $\eta_c \ge 0$, thus $\varepsilon(C) > 0$, see section 5.2 for the definition of $\varepsilon(C)$. Since $\varepsilon_m(u) = 0$ for $1 \le m < i$, the length of C is not less than h. By Lemma 5.1.4, then $k \ge h$. But by Lemma 5.1.4, $q \le h$. Hence by (d) we see $y' < \eta_c$. Therefore $u(a_{cp_c}) \ge u(a_{c,l_c-1}) + n$. This contradicts that C is an r-antichain of u.

If $q = 1$, then $\xi_{c-1} \in \Lambda_h$ and $\eta_c = \eta_{c-1} + 1$. Thus $y \ge \eta_c$ and $y' \le \eta_c - 1$. Since C is an r-antichain in Z, by Lemma 5.1.4 we must have $y = \eta_c$, $y' = \eta_c - 1$ and $j = 1$. This contradicts that $u(a_{cl_c})$ and $u(a_{cp_c})$ have distinct levels.

Thus if $l_c > l_{c-1}$ for some $2 \le c \le h$ we would be led to a contradiction. Therefore (a) is true.

Now we show (b). If $u(a_{c-1,l_c-1}) < u(a_{cl_c})$ for some $2 \le c \le h$, using Lemma 5.4.2, we see that $u(a_{c-1,l_c-1})$ and $u(a_{cl_c})$ have distinct levels. Write

$$u(a_{c-1,l_c-1}) = \xi' + \eta' n, \qquad \xi' \in \Lambda_{k'} \text{ and } \eta' \in \mathbb{Z}.$$

Recall that $\xi_c \in \Lambda_q$. Then

(e) either $k' < q$ and $\eta' \le \eta_c$, or $\eta' < \eta_c$.

Since $l_c \le l_{c-1}$ and $u(a_{c-1,l_{c-1}-1})$, $u(a_{c-1,l_{c-1}})$ have distinct levels, if $q \ge 2$, by Lemma 5.1.4, we have that $k' > q-1$ and $\eta' \ge \eta_{c-1} = \eta_c$, or $\eta' > \eta_{c-1} = \eta_c$. This contradicts (e). Therefore $q = 1$ and $\eta' < \eta_c$. Then $\eta' \ge \eta_{c-1}$ implies that $\eta' = \eta_{c-1} = \eta_c - 1$. Moreover we have $k' > h$ since $u(a_{c-1,l_{c-1}-1})$, $u(a_{c-1,l_{c-1}})$ have distinct levels and $\xi_{c-1} \in \Lambda_h$. Then it is easy to see that the r-antichain D in Z containing $u(a_{c-1,l_c-1})$ does not contain any of $u(a_{c1})$,...,$u(a_{c,l_c-1})$ since none of them is in $\Lambda_1 + (\eta'+1)n$ or $\Lambda_{k'+1} + \eta' n$.

Thus there is an r-antichain E in Z such that E contains $u(a_{cp_c})$ and $u(a_{c-1,p_{c-1}})$ for some $1 \le p_c < l_c$ and $l_c - 1 < p_{c-1} \le \lambda_{c-1}$. As before, write $u(a_{cp_c}) = x + yn$, $u(a_{c-1,p_{c-1}}) = x' + y'n$, where $1 \le x, x' \le n$ and $y, y' \in \mathbb{Z}$. Assume that $x \in \Lambda_j$, $x' \in \Lambda_k$. From the arguments above we see that $y' \le \eta' = \eta_c - 1$ and $y \ge \eta_c$. Thus by Lemma 5.1.4 we must have $y = \eta_c$ and $j = 1$. This is impossible since $u(a_{c,l_c-1})$ and $u(a_{cl_c})$ have distinct levels. We proved (b).

Now we prove (c). Let U be the set consisting of $u(a_{11}), u(a_{12}), ..., u(a_{1,l_1-1})$, and let V be the union of the two sets $\{u(a_{h1}), ..., u(a_{h,\lambda_h-\beta})\}$ and $\{u(a_{c\delta}) \mid h+1 \le c \le r,\ 1 \le \delta \le \lambda_c,\ \text{ and } u(a_{c\delta}) \text{ has the same level as } u(a_{1l_1}) + n\ \}$. Let C be an

r-antichain of u. We shall show that if C contains some element in U, then C contains exactly one element of V, and vice versa.

Let $\alpha \leq \lambda_h - \beta$. Then $u(a_{h\alpha}) > u_{h+1}(e_h)$. Let C be the r-antichain in Z containing $u(a_{h\alpha})$. Assume that $u(a_{1\gamma}) \in C$. We claim that $\gamma < l_1$. Write

$$u(a_{h\alpha}) = \xi + \eta n, \quad u(a_{1\gamma}) = \xi' + \eta' n,$$

where $1 \leq \xi, \xi' \leq n,\ \eta, \eta' \in \mathbb{Z}$. Assume that $\xi \in \Lambda_m$ and $\xi_{h+1} \in \Lambda_{q_{h+1}}$. Then either $m > q_{h+1}$ and $\eta = \eta_{h+1}$, or $\eta > \eta_{h+1}$.

Assume that $m > q_{h+1},\ \eta_{h+1} = \eta$. By Lemma 5.1.4, then $\eta' = \eta - 1$ and $\xi' \in \Lambda_{m'}$ for some $m' > m$ whenever $m < h$. If $m \geq h$, then $\eta' = \eta$. In any case we have $u(a_{1\gamma}) > u(a_{1l_1})$ since $u_{h+1}(e_h) \geq u(a_{1l_1}) + n$. So we have $\gamma < l_1$.

If $\eta > \eta_{h+1}$, then $\eta' \geq \eta_{h+1}$, so we have $u(a_{1\gamma}) > u(a_{1l_1})$ since $u_{h+1}(e_h) \geq u(a_{1l_1}) + n$. Thus we also have $\gamma < l_1$ in this case.

Now suppose that $c > h,\ 1 \leq \delta \leq \lambda_c$ and $u(a_{c\delta})$ has the same level as $u(a_{1l_1})+n$ and D is the r-antichain in Z containing $u(a_{c\delta})$. Assume that $u(a_{1\gamma})$ is in D. We claim that $\gamma < l_1$. We may write

$$u(a_{c\delta}) = \zeta + (\eta_1 + 1)n, \qquad u(a_{1\gamma}) = \xi' + \eta' n,$$

where $1 \leq \zeta, \xi' \leq n,\ \eta' \in \mathbb{Z}$. Assume that $\zeta \in \Lambda_m$. Then either $m \geq c$, $\xi' \in \Lambda_{m-c+1}$ and $\eta' = \eta_1 + 1$; or $m < c$, $\eta' = \eta_1$, and $\xi' \in \Lambda_{m'}$ for some $m' > m$. In any case we have $u(a_{1\gamma}) > u(a_{1l_1})$. Hence $\gamma < l_1$.

Obviously if an r-antichain in Z contains some $u(a_{c\delta})$ for some $c > h$ and $u(a_{c\delta})$ has the same level as $u(a_{1l_1}) + n$, then C does not contains any $u(a_{h\alpha'})$ with $\alpha' \leq \lambda_h - \beta$, since $u(a_{h\alpha'}) > u_{h+1}(e_h) \geq u(a_{c\delta})$.

Now suppose that $\gamma < l_1$ and C is the r-antichain in Z containing $u(a_{1\gamma})$. We claim that either C contains some $u(a_{h\alpha})$ for some $\alpha \leq \lambda_h - \beta$ or C contains some $u(a_{c\alpha})$ for some $c > h$ and $u(a_{c\alpha})$ has the same level as $u(a_{1l_1}) + n$.

As before write $u(a_{1\gamma}) = \xi' + \eta' n$. Assume that $\xi' \in \Lambda_{m'}$ and $\xi_1 \in \Lambda_{q_1}$. Then either $m' > q_1$ and $\eta' = \eta_1$, or $\eta' > \eta_1$, since $u(a_{1\gamma})$ and $u(a_{1l_1})$ have distinct levels. Since $\varepsilon_j(u) = 0$ for $j < i$, by Lemma 5.1.4 we see that the length of C is either h or greater than h. Let $u(a_{h\alpha}) = \xi + \eta n$ be in C, where $\xi \in \Lambda_q$ for some $1 \leq q \leq r$ and $\eta \in \mathbb{Z}$.

Assume that $\eta' > \eta_1$. Suppose that $u(a_{h\alpha}) < u_{h+1}(e_h)$. Then $\eta = \eta_1 + 1$ since $\eta_1 + 1 \geq \eta \geq \eta' > \eta_1$ and $q < q_1$. By Lemma 5.1.4, $q \geq h$ and $q_1 \leq h$. A contradiction. Therefore we have $u(a_{h\alpha}) > u_{h+1}(e_h)$ and we are done in this case.

Assume that $m' > q_1$ and $\eta' = \eta_1$. If $u(a_{h\alpha}) > u_{h+1}(e_h)$ then we are done. Now suppose that $u(a_{h\alpha}) < u_{h+1}(e_h)$. Then we have two cases, (1) $\eta = \eta_1 + 1$ and $q < q_1$, (2) $\eta = \eta_1$ and $q > m' > q_1$. When $\eta = \eta_1 + 1$, since $q < q_1 < m'$, by Lemma 5.1.4 we see that there is $r \geq c > h$ such that $u(a_{c\alpha'}) = \xi'' + \eta'' n \in C$ for some $1 \leq \alpha' \leq \lambda_c$ with $\xi'' \in \Lambda_{q_1}$ and $\eta'' = \eta$. Thus $u(a_{c\alpha'})$ has the same level as $u(a_{1l_1}) + n$. When $\eta = \eta_1$ and $q > m' > q_1$, using Lemma 5.1.4 we see there is some $r \geq c > h$ such that $u(a_{c\alpha'}) = \xi'' + \eta'' n \in C$ for some $1 \leq \alpha' \leq \lambda_c$ with $\xi'' \in \Lambda_{q_1}$ and $\eta'' = \eta + 1$.

Therefore (c) is true.

The proposition is proved.

Proposition 6.3.8. *Let $u, v \in \Gamma_\lambda \cap \Gamma_\lambda^{-1}$ be such that all components of $\varepsilon(u)$ and $\varepsilon(v)$ are non-positive. Assume that*

$$\varepsilon_1(u) = \varepsilon_2(u) = \cdots = \varepsilon_{i-1}(u) = 0$$

and

$$\varepsilon_{in_i}(v) = -1 \text{ and } \varepsilon_{kl}(v) = 0$$

if $(k,l) \neq (i, n_i)$. Then

$$t_u t_v = \sum t_w,$$

where w runs through the set

$$\{w \in \Gamma_\lambda \cap \Gamma_\lambda^{-1} \mid \quad \varepsilon_{ij}(w) = \varepsilon_{ij}(u) - 1 \text{ for some } 1 \leq j \leq n_i, \\ \text{and } \varepsilon_{kl}(w) = \varepsilon_{kl}(u) \text{ if } (k,l) \neq (i,j)\}.$$

Proof. Apply Prop. 6.3.7 and use 1.3 (b) and note that $J_{\Gamma_\lambda\cap\Gamma_\lambda^{-1}}$ is commutative (see Theorem 2.3.2 (b)).

6.4. The factorization formula

Now we can prove the factorization formula. We need a notation. For $x, y \in \Gamma_\lambda \cap \Gamma_\lambda^{-1}$ we define $x * y \in \Gamma_\lambda \cap \Gamma_\lambda^{-1}$ to be the unique element z in $\Gamma_\lambda \cap \Gamma_\lambda^{-1}$ determined by $\varepsilon(z) = \varepsilon(x) + \varepsilon(y)$.

Theorem 6.4.1. *Let $w \in \Gamma_\lambda \cap \Gamma_\lambda^{-1}$ and $w_i \in \Gamma_\lambda \cap \Gamma_\lambda^{-1}$ ($1 \leq i \leq p$) be such that $\varepsilon(w_i) = \varepsilon_i(w)$ (see the beginning of section 6.2 for the definition of $\varepsilon_i(w)$). Then*

$$t_w = t_{w_1} t_{w_2} \cdots t_{w_p}.$$

Proof. We show the result first in the case when w satisfies $\varepsilon_{kl}(w) \geq 0$ for all k, l, and then in general.

Step 1. Assume that all $\varepsilon_{kl}(w) \geq 0$. It is sufficient to prove that if $\varepsilon_k(w) = 0$ for $k = 1, ..., i$ and $\varepsilon(u) = \varepsilon_i(u)$, then $t_u t_w = t_{u*w}$

Let $v \in \Gamma_\lambda \cap \Gamma_\lambda^{-1}$ be such that $\varepsilon_{kl}(v) = 0$ for all pairs (k,l) except $\varepsilon_{i1}(v) = 1$. Let $m \in \mathbb{N}$. According to Prop. 6.3.7, we have

(a) $\qquad t_v^m t_w = \sum_u \theta_u t_u t_w = \sum_u \theta_u t_{u*w},$

where u is in $\Gamma_\lambda \cap \Gamma_\lambda^{-1}$ such that (1) all components of $\varepsilon(u)$ are nonnegative, (2) $\varepsilon_i(u) = \varepsilon(u)$, (3) $\sum_{1\leq j\leq n_i} \varepsilon_{ij}(u) = m$, and (4) θ_u is given by the following formula

$$V(\varepsilon(v))^m = \sum_u \theta_u V(\varepsilon(u)) \qquad \theta_u \in \mathbb{N}.$$

Recall that $V(\varepsilon(u))$ is an irreducible F_λ-module of highest weight $\varepsilon(u)$.

By 1.3 (d), the positivity of the structural coefficients $\gamma_{x,y,z}$'s (see 1.3 (f)) and (a) we get

(b) If $\theta_u \neq 0$, then

$$t_u t_w = t_{x*w}$$

for some $x \in \Gamma_\lambda \cap \Gamma_\lambda^{-1}$ with (1) $\varepsilon_{kl}(x) \geq 0$ for all k, l, (2) $\sum_{1\leq j\leq n_i} \varepsilon_{ij}(x) = \sum_{1\leq j\leq n_i} \varepsilon_{ij}(u) = m$ and (3) $\varepsilon_i(x) = \varepsilon(x)$. Moreover if $u \neq u'$ and $t_{u'}t_w = t_{x'*w}$, then $x \neq x'$.

Clearly θ_u is not zero if and only if (1) all components of $\varepsilon(u)$ are nonnegative, (2) $\varepsilon_i(u) = \varepsilon(u)$, (3) $\sum_{1\leq j\leq n_i} \varepsilon_{ij}(u) = m$.

Let $y \in \Gamma_\lambda \cap \Gamma_\lambda^{-1}$ be such that $\varepsilon_{ij}(y) = k$ for all j and other components of $\varepsilon(y)$ are 0. Then

$$t_y t_w = t_{z*w}$$

for some $z \in \Gamma_\lambda \cap \Gamma_\lambda^{-1}$ with $\varepsilon_i(z) = \varepsilon(z)$ and with $\sum_{1\leq j\leq n_i} \varepsilon_{ij}(z) = kn_i$.

By Prop. 6.3.7, we have $t_v t_y = t_{v*y}$. Hence

$$t_v t_y t_w = t_{v*y} t_w = t_{y'*w}$$

for some $y' \in \Gamma_\lambda \cap \Gamma_\lambda^{-1}$ with $\varepsilon_i(y') = \varepsilon(y')$ and with $\sum_{1\leq j\leq n_i} \varepsilon_{ij}(y') = kn_i + 1$. Thus

$$t_v t_{z*w} = t_{y'*w}.$$

By Prop.6.3.7, this forces that $y = z$. Thus we have

(c) $t_y t_w = t_{y*w}$.

Now we define a total order on the subset M_k of $\Gamma_\lambda \cap \Gamma_\lambda^{-1}$ consisting of all elements u in $\Gamma_\lambda \cap \Gamma_\lambda^{-1}$ with (1) all components of $\varepsilon(u)$ are nonnegative and are less than or equal to k and (2) $\varepsilon_i(u) = \varepsilon(u)$. Let z, z' be in M_k. If $\varepsilon_{ij}(z) = \varepsilon_{ij}(z')$ for $j = l+1, l+2, ..., n_i$ but $\varepsilon_{il}(z) > \varepsilon_{il}(z')$ then we define $z > z'$. This of course introduces a total order on M_k. We have

(d) If $u \in M_k$ and $t_u t_w = t_{x*w}$, then $x \in M_k$.

Otherwise, by (b) we see that $\varepsilon_{i1}(x) > k$. Let $q = kn_i - \varepsilon_{i1}(u) - \cdots - \varepsilon_{in_i}(u)$. Write

$$t_v^q t_u = \sum_{u'} \xi_{u'} t_{u'}, \qquad \xi_{u'} \in \mathbb{N}.$$

By Prop. 6.3.7, $\xi_y \neq 0$. By (c) and Prop. 6.3.7, t_{y*w} appears in $t_v^q t_u t_w$ with nonzero coefficient ξ_y but t_{y*w} appears in $t_v^q t_{x*w}$ with zero coefficient. Thus $t_v^q t_u t_w \neq t_v^q t_{x*w}$. This contradiction shows that x must be in M_k. (For a given element $t = \sum a_z t_z \in J_{\Gamma_\lambda \cap \Gamma_\lambda^{-1}}$, we say that t_z appears in t with coefficient a_z.)

Now we claim that

(e) $t_u t_w = t_{u*w}$ for all u in M_k.

By (c) we see that (e) is true for the maximal element in M_k. Assume that for any $z \in M_k$ with $z > u$ we have $t_z t_w = t_{z*w}$. By (c) we may assume that u is not maximal and we can find $1 \leq j < n_i$ such that $\varepsilon_{ij}(u) > \varepsilon_{i,j+1}(u) = \cdots = \varepsilon_{in_i}(u)$.

We have $t_u t_w = t_{x*w}$ for some $x \in M_k$. Suppose that $u \neq x$. Since for $x > u$ we have $t_x t_w = t_{x*w}$, using (b) we get $x < u$.

Denote by τ_{il} the element in $\mathbb{Z}^{n_1} \times \cdots \times \mathbb{Z}^{n_p}$ whose (i,l)-component is 1 and other components are 0. Consider the product

$$t_v t_u t_w = t_v t_{x*w} = \sum_z \eta_z t_{z*w}, \quad \eta_z \in \mathbb{N}.$$

Let $u' \in M_k$ be determined by $\varepsilon(u') = \varepsilon(u) + \tau_{i,j+1}$, then $t_{u'}$ appears in $t_v t_u$ with coefficient 1. By induction hypothesis, $t_{u'*w}$ appears in $t_v t_{x*w}$ with nonzero coefficient. According to Prop. 6.3.7, this forces that $\varepsilon(x) = \varepsilon(u) + \tau_{i,j+1} - \tau_{i,n_i}$ since $x < u$.

If $\varepsilon_{i1}(u) < k$ or there is $j' > j$ such that $\varepsilon_{ij'}(u) > \varepsilon_{i,j'+1}(u)$, then $t_{v'}$ appears in $t_v t_u$ with coefficient 1, where $v' \in \Gamma_\lambda \cap \Gamma_\lambda^{-1}$ with $\varepsilon(v') = \varepsilon(u) + \tau_{i1}$ or $\varepsilon(v') = \varepsilon(u) + \tau_{i,j'+1}$. Obviously $v' \in M_k$ and $v' > u$, by induction hypothesis, then $t_{v'*w}$ appears in $t_v t_{x*w}$ with coefficient 1. But this is impossible since by Prop. 6.3.7, if $x' \neq u' * w$ and $t_{x'}$ appears in $t_v t_{x*w}$ with non zero coefficient, then $\varepsilon_{in_i}(x') = \varepsilon_{in_i}(u) - 1$.

Thus we must have

$$\varepsilon_{i1}(u) = \cdots = \varepsilon_{ij}(u) = k > \varepsilon_{i,j+1}(u) = \cdots = \varepsilon_{in_i}(u).$$

Define $\psi(\sum k_z t_z) = \sum k_z$.

If $n_i - j > 2$ or $n_i - j = 2$ but $\varepsilon_{ij}(u) > \varepsilon_{i,j+1} + 1$, then $\psi(t_v t_u t_w) = 2$ but $\psi(t_v t_{x*w}) \geq 3$. This is impossible. If $n_i = j + 1$ we have $x = u$ since $\varepsilon(x) = \varepsilon(u) + \tau_{i,j+1} - \tau_{i,n_i}$.

If $n_i = j + 2$ and $\varepsilon_{ij}(u) = k = \varepsilon_{i,j+1}(u) + 1$, we can prove directly that the assertion (e) is true in this case. In fact, from $t_y t_w = t_{y*w}$ and by considering $t_v t_{y'} t_w$ we see that $t_{y'} t_w = t_{y'*w}$ if $\varepsilon(y') = \varepsilon(y) - \tau_{in_i}$. Then by considering $t_v t_{y''} t_w$ we see that $t_{y''} t_w = t_{y''*w}$ if $\varepsilon(y') = \varepsilon(y) - 2\tau_{in_i}$ and $k \geq 2$. Thus by considering $t_v t_u t_w$ we see that $t_u t_w = t_{u*w}$ in this case.

The theorem is proved when all $\varepsilon_{kl}(w) \geq 0$.

Step 2. Now $w \in \Gamma_\lambda \cap \Gamma_\lambda^{-1}$ is arbitrary. We can find $q \in \mathbb{N}$ such that $\varepsilon_{kl}(w) + qr_k \geq 0$ for all k, l. Let $u = \omega^{qn} w$ and $u_i \in \Gamma_\lambda \cap \Gamma_\lambda^{-1}$ with $\varepsilon(u_i) = \varepsilon_i(u)$. Then $\varepsilon_{kl}(u) = \varepsilon_{kl}(w) + qr_k \geq 0$ for all k, l. By step 1 we have

(f) $t_u = t_{u_1} t_{u_2} \cdots t_{u_p}$.

Obviously we have

(g) $t_w = t_u t_{\omega^{-qn} w_\lambda} = t_{u_1} t_{u_2} \cdots t_{u_p} t_{\omega^{-qn} w_\lambda}$.

Since $\omega^{qn} w_\lambda \in \Gamma_\lambda \cap \Gamma_\lambda^{-1}$ and $\varepsilon_{kl}(\omega^{qn} w_\lambda) = qr_k \geq 0$, by step 1, we have

(h) $t_{\omega^{qn} w_\lambda} = t_{\theta_1} t_{\theta_2} \cdots t_{\theta_p}$, where $\theta_k \in \Gamma_\lambda \cap \Gamma_\lambda^{-1}$ is determined by $\varepsilon(\theta_k) = \varepsilon_k(\omega^{qn} w_\lambda)$.

Using 1.3 (b) and Theorem 2.3.2 (b) we get

(i) $t_{\omega^{-qn} w_\lambda} = t_{\theta_1^{-1}} t_{\theta_2^{-1}} \cdots t_{\theta_p^{-1}}$.

Obviously we have $t_{\omega^{qn} w_\lambda} t_{\omega^{-qn} w_\lambda} = t_{w_\lambda}$. Using 1.3 (c) and 1.3 (f) we see that $t_{\theta_k} t_{\theta_k^{-1}} = t_{w_\lambda}$ for all k. Thus we have

(j) $t_{\theta_k}^{-1} = t_{\theta_k^{-1}}$.

According to step 1 we have

(k) $t_{\theta_1}t_{\theta_2}\cdots t_{\theta_{k-1}}t_{u_k}t_{\theta_{k+1}}\cdots t_{\theta_p} = t_{v_k}$ for all k, where $v_k = \theta_1 * \theta_2 * \cdots * \theta_{k-1} * u_k * \theta_{k+1} * \cdots * \theta_p$.

Note that $\varepsilon(v_k\omega^{-qn}) = \varepsilon(w_k) = \varepsilon(u_k) - \varepsilon(\theta_k)$ for all k. Using Theorem 5.2.6 (b), we get $v_k\omega^{-qn} = w_k$ for all k. Then obviously we have $t_{v_k}t_{\omega^{-qn}w_\lambda} = t_{w_k}$. Using (f)-(k) we get

$$\begin{aligned} t_w &= t_u t_{\omega^{-qn}w_\lambda} \\ &= \textstyle\prod_{k=1}^{p}(t_{\theta_1}t_{\theta_2}\cdots t_{\theta_{k-1}}t_{u_k}t_{\theta_{k+1}}\cdots t_{\theta_p}t_{\theta_1^{-1}}t_{\theta_2^{-1}}\cdots t_{\theta_p^{-1}}) \\ &= \textstyle\prod_{k=1}^{p} t_{v_k}t_{\omega^{-qn}w_\lambda} \\ &= t_{w_1}t_{w_2}\cdots t_{w_p}. \end{aligned}$$

The theorem is proved.

CHAPTER 7

A Multiplication Formula in $J_{\Gamma_\lambda\cap\Gamma_\lambda^{-1}}$

Let λ be as in section 2.2 and n_i be as in section 4.1. Then $GL_{n_i}(\mathbb{C})$ is the ith reductive component of F_λ. Let $\varepsilon=(\varepsilon_{11},...,\varepsilon_{pn_p})$ and $\varepsilon'=(\varepsilon'_{11},...,\varepsilon'_{pn_p})$ be two elements in $\mathrm{Dom}(F_\lambda)$ such that $\varepsilon_{kl}=\varepsilon'_{kl}=0$ whenever $k\neq i$ and $\varepsilon_{i1}=2,\ \varepsilon_{i2}=\cdots=\varepsilon_{in_i}=0$. Let $V(\varepsilon)$ and $V(\varepsilon')$ be two irreducible F_λ-modules of highest weight ε and ε' respectively. Then we have a simple formula for the product $V(\varepsilon)V(\varepsilon')$ in R_{F_λ} (see 4.2 (g)). In this chapter we will establish a multiplication formula in $J_{\Gamma_\lambda\cap\Gamma_\lambda^{-1}}$ that is corresponding to the formula for $V(\varepsilon)V(\varepsilon')$, see Theorem 7.2.2. To do this we compute some product $\tilde{T}_u\tilde{T}_v$ in section 7.1. Then in section 7.2 we prove our formula. In Chapter 8, using this formula and the factorization formula (Theorem 6.4.1) we show that Conjecture 2.3.3 is true.

7.1. A computation for some $\tilde{T}_u\tilde{T}_v$

Let u,v be in $\Gamma_\lambda\cap\Gamma_\lambda^{-1}$ such that $\varepsilon(u)=\varepsilon_i(u)$ and $\varepsilon_{i1}(v)=2$ and $\varepsilon_{kl}(v)=0$ whenever $(k,l)\neq(i,1)$. In this section we will compute the product $\tilde{T}_u\tilde{T}_v$ provided that all components of $\varepsilon(u)$ are non-negative. We show that $f_{u,v,w}$ have degree less than or equal to $a(w_\lambda)$ for the u,v and all $w\in W$. Keep the notation in Chapter 5. Let v_{i1} be the u_{i1} in section 5.5. We have

Lemma 7.1.1. *Let $1\leq k\leq r$. Suppose that $e_{k-1}+1\leq q\leq e_k-2$. Then*
(a) $\tilde{T}_{s_q}\tilde{T}_{v_{i1}}=\tilde{T}_{v_{i1}}\tilde{T}_{s_q}$,
(b) $\tilde{T}_{s_q}^{-1}\tilde{T}_{v_{i1}}=\tilde{T}_{v_{i1}}\tilde{T}_{s_q}^{-1}$.

Proof. (a) According to section 5.5 we have $v_{i1}=\tau_{\lambda_1}s(e_1,e_2,...,e_h)$. Using this we see easily that $s_qv_{i1}=v_{i1}s_q$ and $l(s_qv_{i1})=1+l(v_{i1})$. Thus

$$\tilde{T}_{s_q}\tilde{T}_{v_{i1}}=\tilde{T}_{s_qv_{i1}}=\tilde{T}_{v_{i1}s_q}=\tilde{T}_{v_{i1}}\tilde{T}_{s_q}.$$

We proved (a). (b) follows from (a).

7.1.2. Let u be in $\Gamma_\lambda\cap\Gamma_\lambda^{-1}$ such that (1) $\varepsilon_i(u)=\varepsilon(u)$ and (2) all components of $\varepsilon(u)$ are nonnegative. Let $v=v_{i1}^2w_\lambda$. By 5.5 (d-e), we have $v\in\Gamma_\lambda\cap\Gamma_\lambda^{-1}$, $\varepsilon_{kl}(v)=0$ if $(k,l)\neq(i,1)$ and $\varepsilon_{i1}(v)=2$. Now we compute $\tilde{T}_u\tilde{T}_v$. Recall that we have set $e_k=\lambda_1+\cdots+\lambda_k$, $e_0=0$ and $r_i=h$. By the construction in section 5.3 we see

$(*)$ $u(a)=v(a)=w_\lambda(a)$ if either $e_h+1\leq a\leq n$ or $a=a_{kl}$ for some k with $l>\lambda_h$.

According to 5.5 (g) we get

$$\begin{aligned} v_{i1} = \ & \omega s_{e_1-2}s_{e_1-3}\cdots s_1 s_0 \hat{s}_{e_2-1}s_{e_2-2}\cdots s_{e_1} \\ & \times \hat{s}_{e_{h-1}-1}s_{e_{h-1}-2}\cdots s_{e_{h-2}}\hat{s}_{e_h-1}s_{e_h-2}\cdots s_{e_{h-1}} \\ & \times s_{n-1}s_{n-2}\cdots s_{e_h} \end{aligned}$$

Set

$$\delta = s_{n-1}\cdots s_{e_h}.$$

Let $1 \leq j \leq h-1$. Suppose that $z_k \leq s_{e_k-2}\cdots s_{e_{k-1}}$ for $k = 1, 2, ..., j$. Using 2.1.3 (f) we see

(a) $u\omega z_1 z_2\cdots z_j s_q \leq u\omega z_1 z_2 \cdots z_j$ for all $e_j \leq q \leq e_{j+1}-2$.

According to $(*)$ and Lemma 6.1.3, we have

$$\text{(b)} \qquad \tilde{T}_u\tilde{T}_{v_{i1}} = \sum_{\substack{z_{1,1}\leq s_{e_1-2}\cdots s_{e_0} \\ \cdots \\ z_{h,1}\leq s_{e_h-2}\cdots s_{e_{h-1}}}} \left(\prod_{k=1}^{h} \xi^{\lambda_k-1-l(z_{k,1})}\right)\tilde{T}_{u\omega z_{1,1}\cdots z_{h,1}}\tilde{T}_\delta.$$

Using Lemma 5.3.2 and 2.1.3 (f) we see that

$$l(u\omega z_{1,1}\cdots z_{h,1}\delta) = l(u\omega z_{1,1}\cdots z_{h,1}) + l(\delta).$$

Thus we have

$$\tilde{T}_{u\omega z_{1,1}\cdots z_{h,1}}\tilde{T}_\delta = \tilde{T}_{u\omega z_{1,1}\cdots z_{h,1}\delta}.$$

Let

$$u_1' = u\omega z_{1,1}\cdots z_{h,1}\delta.$$

Then for $1 \leq k \leq h-1$ we have $u_1'(e_k) = u(a_{k+1,p_{k+1,1}})$ for some $1 \leq p_{k+1,1} \leq \lambda_{k+1}$ and $u_1'(e_h) = u(a_{1,p_{1,1}}) + n$ for some $1 \leq p_{1,1} \leq \lambda_1$. Thus

$$z_{k,1} = s_{e_{k-1}+p_{k,1}-2}\cdots s_{e_{k-1}}y_{k,1}$$

for some $y_{k,1} \leq s_{e_k-2}s_{e_k-3}\cdots s_{e_{k-1}+p_{k,1}}$. We have

$$l(u_1'y_{1,1}^{-1}\cdots y_{h,1}^{-1}) = l(u_1') + l(y_{1,1}^{-1}) + \cdots + l(y_{h,1}^{-1}).$$

Let $u_1 = u_1'y_{1,1}^{-1}\cdots y_{h,1}^{-1}$. Then

$$\tilde{T}_{u_1'} = \tilde{T}_{u_1}\tilde{T}^{-1}_{y_{1,1}^{-1}}\cdots\tilde{T}^{-1}_{y_{h,1}^{-1}}.$$

Note that $l(z_{k,1}) = l(y_{k,1}) + p_{k,1} - 1$. Thus we have

$$\text{(c)} \quad \tilde{T}_u\tilde{T}_{v_{i1}} = \sum_{\substack{z_{1,1}\leq s_{e_1-2}\cdots s_{e_0} \\ \cdots \\ z_{h,1}\leq s_{e_h-2}\cdots s_{e_{h-1}}}} \left(\prod_{k=1}^{h} \xi^{\lambda_k-p_{k,1}}\right)\tilde{T}_{u_1}(\xi^{-l(y_{1,1})}\tilde{T}^{-1}_{y_{1,1}^{-1}})\cdots(\xi^{-l(y_{h,1})}\tilde{T}^{-1}_{y_{h,1}^{-1}}).$$

Moreover we have

$$\text{(d)}\quad u_1(a) = \begin{cases} u(a) & \text{if } e_{k-1}+1 \le a \le e_{k-1}+p_{k,1}-1, \quad k=1,2,...,h,\\ u(a+1) & \text{if } e_{k-1}+p_{k,1} \le a \le e_{k-1}-1, \quad k=1,2,...,h,\\ u(a_{k+1,p_{k+1,1}}) & \text{if } a=e_k, \quad k=1,2,...,h-1,\\ u(a_{1,p_{1,1}})+n & \text{if } a=e_h,\\ u(a) & \text{if } e_h < a \le n. \end{cases}$$

In particular we have

(e) $u_1(a) > u_1(b)$ if $e_k+1 \le a < b \le e_{k+1}-1$ for some $0 \le k \le r-1$.

For any $w \in W$ we set $w(a_{k0}) = \infty$ and $w(a_{k,\lambda_k+1}) = -\infty$ for all k.

For $1 \le k \le h$ choose $0 \le q_{k,2} \le \lambda_k - 1$ such that

$$u_1(a_{k,q_{k,2}}) > u_1(e_k) \ge u_1(a_{k,q_{k,2}+1}).$$

Using 2.1.3 (f) we get

$$\text{(f)}\qquad l(u_1\omega\prod_{k=1}^{h}(s_{e_k-2}\cdots s_{e_{k-1}+q_{k,2}})) = l(u_1) + \sum_{k=1}^{h}(\lambda_k - q_{k,2} - 1).$$

Let $u_2' = u_1\omega\prod_{k=1}^{h}(s_{e_k-2}\cdots s_{e_{k-1}+q_{k,2}})$. Then we have

$$\text{(g)}\qquad \begin{aligned} u_2'(a) &= \begin{cases} u_1(a+1) & \text{if } e_{k-1} \le a \le e_{k-1}+q_{k,2}-1, \quad k=1,2,...,h,\\ u_1(e_k) & \text{if } a = e_{k-1}+q_{k,2}, \quad k=1,2,...,h,\\ u_1(a) & \text{if } e_{k-1}+q_{k,2} < a < e_k, \quad k=1,2,...,h,\\ u_1(a+1) & \text{if } e_h \le a \le n-1. \end{cases}\\ u_2'\omega^{-1} &= \begin{cases} u_1(a) & \text{if } e_{k-1}+1 \le a \le e_{k-1}+q_{k,2}, \quad k=1,2,...,h,\\ u_1(e_k) & \text{if } a = e_{k-1}+q_{k,2}+1, \quad k=1,2,...,h,\\ u_1(a-1) & \text{if } e_{k-1}+q_{k,2}+1 < a \le e_k, \quad k=1,2,...,h,\\ u_1(a) & \text{if } e_h+1 \le a \le n. \end{cases} \end{aligned}$$

As a consequence we have

(h) $u_2'(a) = u(a+1)$ if $e_h \le a \le n-1$ and $u_2's_q \le u_2'$ if $e_{k-1} \le q \le e_{k-1}+q_{k,2}-1$ for some $1 \le k \le h$. Moreover, $u_2'(a) > u_2'(b)$ if $e_k \le a < b \le e_{k+1}-1$ for some $0 \le k \le r-1$.

Thus we have

$$\text{(i)}\qquad \tilde{T}_{u_1}\tilde{T}_{v_{i1}} = \tilde{T}_{u_2'}\tilde{T}((\prod_{k=1}^{h} s_{e_{k-1}+q_{k,2}-1}\cdots s_{e_{k-1}})\delta),$$

and (here we need Lemma 6.1.3 and recall that $\tilde{T}(w)$ is set to be $\tilde{T}_w$)

$$\text{(j)}\qquad \tilde{T}_{u_1}\tilde{T}_{v_{i1}} = \sum_{\substack{z_{1,2}\le s_{e_0+q_{1,2}-1}\cdots s_{e_0}\\ \cdots\\ z_{h,2}\le s_{e_{h-1}+q_{h,2}-1}\cdots s_{e_{h-1}}}} (\prod_{k=1}^{h}\xi^{q_{k,2}-l(z_{k,2})})\tilde{T}(u_2' z_{1,2}\cdots z_{h,2})\tilde{T}_\delta.$$

We may check that

$$l(u_2'z_{1,2}\cdots z_{h,2}\delta) = l(u_2'z_{1,2}\cdots z_{h,2}) + l(\delta).$$

Therefore we have

$$\tilde{T}_{u_2'z_{1,2}\cdots z_{h,2}}\tilde{T}_\delta = \tilde{T}_{u_2'z_{1,2}\cdots z_{h,2}\delta}.$$

Assume that

$$u_2'z_{1,2}\cdots z_{h,2}\delta(e_k) = u_2'\omega^{-1}(a_{k+1,p_{k+1,2}})$$

for $k = 1, ..., h-1$ and

$$u_2'z_{1,2}\cdots z_{h,2}\delta(e_h) = u_2'\omega^{-1}(a_{1,p_{1,2}}) + n.$$

Then

$$z_{k,2} = s_{e_{k-1}+p_{k,2}-2}\cdots s_{e_{k-1}}y_{k,2}$$

for some $y_{k,2} \le s_{e_{k-1}+q_{k,2}-1}s_{e_{k-1}+q_{k,2}-2}\cdots s_{e_{k-1}+p_{k,2}}$. We have

$$l(u_2'z_{1,2}\cdots z_{h,2}\delta y_{1,2}^{-1}\cdots y_{h,2}^{-1}) = l(u_2'z_{1,2}\cdots z_{h,2}\delta) + l(y_{1,2}^{-1}) + \cdots + l(y_{h,2}^{-1}).$$

Let $u_2 = u_2'z_{1,2}\cdots z_{h,2}\delta y_{1,2}^{-1}\cdots y_{h,2}^{-1}$. Then

$$\tilde{T}_{u_2'z_{1,2}\cdots z_{h,2}\delta} = \tilde{T}_{u_2}\tilde{T}^{-1}_{y_{1,2}^{-1}}\cdots\tilde{T}^{-1}_{y_{h,2}^{-1}}.$$

Note that $l(z_{k,2}) = l(y_{k,2}) + p_{k,2} - 1$. Thus we have

$$\tilde{T}_{u_1}\tilde{T}_{v_{i1}} = \sum_{\substack{z_{1,2}\le s_{e_0+q_{1,2}-1}\cdots s_{e_0}\\ \cdots\\ z_{h,2}\le s_{e_{h-1}+q_{h,2}-1}\cdots s_{e_{h-1}}}} (\prod_{k=1}^{h}\xi^{q_{k,2}-p_{k,2}+1})\tilde{T}_{u_2}(\xi^{-l(y_{1,2})}\tilde{T}^{-1}_{y_{1,2}^{-1}})\cdots(\xi^{-l(y_{h,2})}\tilde{T}^{-1}_{y_{h,2}^{-1}}).$$

Moreover we have

(k) $u_2(e_k) = u_2'\omega^{-1}(a_{k+1,p_{k+1,2}})$ for $k = 1, ..., h-1$ and $u_2(e_h) = u_2'\omega^{-1}(a_{1p_{1,2}}) + n$.

(l) $u_2(a) = u(a)$ for all $e_h < a \le n$, and $u_2(a) > u_2(b)$ if $e_k + 1 \le a < b \le e_{k+1} - 1$ for some $k = 0, 1, 2, ..., h-1$.

From the above discussion and using Lemma 7.1.1 we see that

(m)

$$\tilde{T}_u\tilde{T}_v = \sum_{u_2} f'_{u,v,u_2}\tilde{T}_{u_2}\prod_{m=2}^{1}((\xi^{-l(y_{1,m})}\tilde{T}^{-1}_{y_{1,m}^{-1}})\cdots(\xi^{-l(y_{h,m})}\tilde{T}^{-1}_{y_{h,m}^{-1}}))\tilde{T}_{w_\lambda}, \qquad f'_{u,v,u_2}\in\mathcal{A},$$

where $f'_{u,v,u_2} = \prod_{k=1}^{h}\xi^{\lambda_k - p_{k,1}+q_{k,2}-p_{k,2}+1}$ has degree

$$D = \sum_{k=1}^{h}(\lambda_k - p_{k,1} + q_{k,2} - p_{k,2} + 1).$$

Using Lemma 5.4.4, (d) and Lemma 5.1.1 (c) we see that

(n) $q_{k,2} \le p_{k+1,1} - 1$ for $k = 1, 2, ..., h-1$, and $q_{h,2} \le p_{1,1} - 1$.

Since $u(a_{k+1,p_{k+1,1}}) \le u_2'\omega^{-1}(a_{k,p_{k,2}})$ for $k = 1, ..., h-1$ and $u(a_{1,p_{1,1}}) + n \le u_2'\omega^{-1}(a_{h,p_{h,2}})$, we have

(o) Let $1\le k\le h-2$. We have

$u_2(\Lambda_k)=(u(\Lambda_k)-\{u(a_{k,p_{k,1}}),\ u_2'\omega^{-1}(a_{k,p_{k,2}})\})\bigcup\{u(a_{k+1,p_{k+1,1}}),\ u_2'\omega^{-1}(a_{k+1,p_{k+1,2}})\}$.

If $u(a_{k+2,p_{k+2,1}})>u(a_{k+1,p_{k+1,1}})$, then

$$1\le p_{k+1,2}\le q_{k+1,2}+1\le p_{k+1,1}.$$

In this case we have

$$u_2'\omega^{-1}(a_{k+1,p_{k+1,2}})\ge u(a_{k+2,p_{k+2,1}})>u(a_{k+1,p_{k+1,1}}),$$

and $1\le p_{k+1,2}\le q_{k+1,2}$ or $p_{k+1,2}=q_{k+1,2}+1$. Using (d), (g) and Lemma 5.4.4 we see that

(o1) among $u_2(a_{k,1}),...,u_2(a_{k,\lambda_k-1})$, at least $\lambda_k-p_{k+1,2}$ of them are smaller than $u_2(a_{k\lambda_k})$.

Similarly, if $u(a_{k+2,p_{k+2,1}})<u(a_{k+1,p_{k+1,1}})$, then we have

(o2) among $u_2(a_{k,1}),...,u_2(a_{k,\lambda_k-1})$, at least $\lambda_k-p_{k+1,2}-1$ of them are smaller than $u_2(a_{k\lambda_k})$, and $q_{k+1,2}\le p_{k+2,1}-2$.

Similarly for $k=h-1,h$ we have

(o3) either among $u_2(a_{k,1}),...,u_2(a_{k,\lambda_k-1})$, at least $\lambda_k-p_{k+1,2}$ of them are smaller than $u_2(a_{k\lambda_k})$, or among $u_2(a_{k,1}),...,u_2(a_{k,\lambda_k-1})$, at least $\lambda_k-p_{k+1,2}-1$ of them are smaller than $u_2(a_{k\lambda_k})$ and $q_{k+1,2}\le p_{k+2,1}-2$, where we set $p_{h+k,2}=p_{k,2}$ and $q_{h+k,2}=q_{k,2}$.

Let x be the element in u_2W_λ of minimal length (see 6.1 for the definition of W_λ). Then $u_2=xy$ for some $y\in W_\lambda$. Using (n) and (o1-o3) we get

(p) $D+l(y)\le l(w_\lambda)$.

We also have

(q) $\tilde T^{-1}_{y^{-1}_{k,m}}\tilde T_{w_\lambda}=\tilde T_{y^{-1}_{k,m}w_\lambda}$ and $l(y^{-1}_{k,m}w_\lambda)=l(w_\lambda)-l(y^{-1}_{k,m})$ for all k,m.

(r) For any $w\in W,\ s\in S$ we have

$$\xi^{-1}\tilde T_s^{-1}\tilde T_w=\begin{cases}\xi^{-1}\tilde T_{sw} & \text{if } ws\le w\\ \xi^{-1}\tilde T_{sw}-\tilde T_w & \text{if } ws\ge w.\end{cases}$$

Thus we see in the expression $\tilde T_u\tilde T_v=\sum_w f_{u,v,w}\tilde T_w$, the degree L of $f_{u,v,w}$ is at most $D+l(y)$. Using (p) we see that

$$L\le a(w_\lambda)=l(w_\lambda).$$

Using (q) and (r) we see that

(s) If $L=a(w_\lambda)$ then $y_{k,m}$ are the neutral element of W for all k,m, here we set $p_{h+1,1}=p_{1,1}$.

7.2. A multiplication formula

In this section we give a multiplication formula in $J_{\Gamma\cap\Gamma^{-1}}$, based on the computation in section 7.1.

Theorem 7.2.1. *Let u,v be in $\Gamma_\lambda\cap\Gamma_\lambda^{-1}$ such that all components of $\varepsilon(u)$ are non-negative, $\varepsilon_i(u)=\varepsilon(u)$, $\varepsilon_{i1}(v)=2$ and other components of $\varepsilon(v)$ are 0. Then*

$$t_ut_v=\sum_w t_w,$$

where w runs through the set

$$\{w \in \Gamma_\lambda \cap \Gamma_\lambda^{-1} \quad | \ \varepsilon(w) = \varepsilon(u) + \tau_{ij_1} + \tau_{ij_2} \text{ for some } 1 \le j_2 \le j_1 \le n_i \text{ and } \varepsilon(u) + \tau_{ij_1} \in Dom(F_\lambda)\},$$

see the proof of Theorem 6.4.1 for the definition of τ_{ij}.

Proof. Keep the notation in 7.1.2.

Assume that $\varepsilon(u)+\tau_{ij_1} \in \mathrm{Dom}(F_\lambda)$ and $w \in \Gamma_\lambda \cap \Gamma_\lambda^{-1}$, $\varepsilon(w) = \varepsilon(u)+\tau_{ij_1}+\tau_{ij_2}$. For all $1 \le k \le h$, set

$$z_{k,1} = s_{e_{k-1}+j_1-2} \cdots s_{e_{k-1}+1} s_{e_{k-1}}$$

and

$$z_{k,2} = s_{e_{k-1}+j_2-2} \cdots s_{e_{k-1}+1} s_{e_{k-1}}.$$

From the construction in section 5.3 and the arguments in 7.1.2 we see that $f_{u,v,w}$ has degree $a(w_\lambda)$ and its leading coefficient is 1.

Now suppose that $w \in \Gamma_\lambda \cap \Gamma_\lambda^{-1}$ and $f_{u,v,w}$ has degree $a(w_\lambda)$. Let x, y be in $\Gamma_\lambda \cap \Gamma_\lambda^{-1}$ such that $\varepsilon_{i1}(x) = 1$, $\varepsilon_{i1}(y) = \varepsilon_{i2}(y) = 1$ and all other components of $\varepsilon(x)$ and $\varepsilon(y)$ are 0. According to Prop. 6.3.7 we have $t_x^2 = t_y + t_v$. Using Prop. 6.3.7 and the positivity 1.3 (f) we see that

$$(\star) \qquad \varepsilon(w) = \varepsilon_i(w) \text{ and } \sum_{1 \le j \le n_i} \varepsilon_{ij}(w) = \sum_{1 \le j \le n_i} \varepsilon_{ij}(u) + 2.$$

Moreover, all $p_{k,m}$ $(1 \le k \le h,\ m = 1, 2)$ are not greater than λ_h. By Lemma 5.4.4, 7.1.2 (n-p), $u(a_{k,p_{k+1,1}-1}) > u(a_{k+1,p_{k+1,1}})$ for $k = 1, 2, ..., h-1$ and $u(a_{h,p_{h+1,1}-1}) > u(a_{1,p_{1,1}} + n)$. We claim that $p_{1,1} = p_{2,1} = \cdots = p_{h,1}$.

Otherwise, $p_{k,1} \ne p_{k+1,1}$ for some $1 \le k \le h$ (recall that $p_{h+k,1} = p_{k,1}$). By $(\star)$ and the construction in section 5.3, $w(\Lambda_k)$ can not contain both $u(a_{k,p_{k+1,1}})$ and $u(a_{k+1,p_{k+1,1}})$ (we set $u(a_{h+1,p_{h+1,1}}) = u(a_{1,p_{1,1}}) + n$). By the arguments in 7.1.2 (o) we see that

$(*)$ $u(a_{k+1,p_{k+1,1}})$ is in $w(\Lambda_{k-1})$ (we set $\Lambda_0 = \Lambda_h$).

Using the construction in section 5.3, we must have $p_{k,1} = p_{k+1,1}$. A contradiction, so the assumption $p_{k,1} \ne p_{k+1,1}$ is not true.

In a complete similar way we see that $p_{1,2} = p_{2,2} = \cdots = p_{h,2}$. Since $w \in \Gamma_\lambda \cap \Gamma_\lambda^{-1}$ and $\varepsilon(w) = \varepsilon_i(w)$, we must have $1 \le p_{1,1}, p_{1,2} \le n_i$. From the arguments in 7.1.2 we have $p_{1,2} \le p_{1,1}$. Thus $\varepsilon(w) = \varepsilon(u) + \tau_{ij_1} + \tau_{ij_2}$ for $1 \le j_2 = p_{1,2} \le j_1 = p_{1,1} \le n_i$.

The theorem is proved.

Theorem 7.2.2. *Let u, v be in $\Gamma_\lambda \cap \Gamma_\lambda^{-1}$ such that $\varepsilon_i(u) = \varepsilon(u)$, all components of $\varepsilon(u)$ are non-negative, $\varepsilon_{i1}(v) = \varepsilon_{i2}(v) = 1$ and other components of $\varepsilon(v)$ are 0. Then*

$$t_u t_v = \sum_w t_w,$$

where w runs through the set

$$\{w \in \Gamma_\lambda \cap \Gamma_\lambda^{-1} \mid \varepsilon(w) = \varepsilon(u) + \tau_{ij_1} + \tau_{ij_2} \text{ for some } 1 \le j_2 < j_1 \le n_i,\ \}.$$

Proof. Let x, y be in $\Gamma_\lambda \cap \Gamma_\lambda^{-1}$ such that $\varepsilon_{i1}(x) = 1$, $\varepsilon_{i1}(y) = 2$ and all other components of $\varepsilon(x), \varepsilon(y)$ are 0. According to Prop. 6.3.7 we have $t_x^2 = t_y + t_v$. Now considering $t_u t_x^2$ and using Prop.6.3.7 and Theorem 7.2.1 we see that the required result is true.

CHAPTER 8

The Based Rings $J_{\Gamma_\lambda\cap\Gamma_\lambda^{-1}}$ and $J_{\mathbf{c}}$

In this chapter we will prove Conjecture 2.3.3 using the formulas in Chapters 6 and 7. This completes our proof of Lusztig Conjecture for type $\tilde{A}_{n-1}$. In section 8.1 we give some lemmas about multiplication in $J_{\Gamma_\lambda\cap\Gamma_\lambda^{-1}}$. In section 8.2 we give a proof for Conjecture 2.3.3 and give a summary of our proof of Lusztig Conjecture on based ring for type $\tilde{A}_{n-1}$. Also we give a few comments about a possible geometric realization of $J_{\Gamma_\lambda\cap\Gamma_\lambda^{-1}}$. In section 8.3 we consider the affine Weyl group associated with the projective linear group $PGL_n(\mathbb{C})$. For the affine Weyl group we show that Lusztig Conjecture on based ring should have a weak form since the algebraic group $PGL_n(\mathbb{C})$ is not simply connected. In section 8.4 we show that Lusztig Conjecture on based ring is true for the extended affine Weyl group associated with the special linear group $SL_n(\mathbb{C})$.

8.1. Some lemmas

In this section we establish some lemmas about multiplication in $J_{\Gamma_\lambda\cap\Gamma_\lambda^{-1}}$. Let N be the subset of $\Gamma_\lambda\cap\Gamma_\lambda^{-1}$ consisting of all elements u in $\Gamma_\lambda\cap\Gamma_\lambda^{-1}$ with $\varepsilon(w)=\varepsilon_i(w)$. Let w_j ($1\le j\le n_i$) be in N such that
(1)$\varepsilon_{i1}(w_j)=\varepsilon_{i2}(w_j)=\cdots=\varepsilon_{ij}(w_j)=1$,
(2) other components of $\varepsilon(w_j)$ are 0.
Then w_j is corresponding to the (i,j)-th fundamental weight of F_λ. Using the bijection $\varepsilon:\Gamma_\lambda\cap\Gamma_\lambda^{-1}\to$Dom$(F_\lambda)$ we shall identify N with the set $\mathbb{Z}^{n_i}_{\rm dom}$ (see 4.1 for the definition of $\mathbb{Z}^{n_i}_{\rm dom}$). Under the identification, we have

$$w=(\varepsilon_{i1}(w),\varepsilon_{i2}(w),...,\varepsilon_{in_i}(w))$$

if $w\in N$.

Lemma 8.1.1. *Let $w\in N$. Then we have*

$$t_{w_{n_i}}t_w=t_{w_{n_i}*w}.$$

See section 6.4 for the definition of $w_{n_i}*w$.

Proof. When all components of $\varepsilon(w)$ are nonnegative, it is entirely similar to the proof for Theorem 6.4.1. Here we need consider $t_{w_{n_i}}t^m_{w_1}$ first. Thus the lemma is true if all components of $\varepsilon(w)$ are nonnegative.

Note that $\omega^{kn}w$ is in $\Gamma_\lambda\cap\Gamma_\lambda^{-1}$. In general we can find a positive integer k such that all components of $\varepsilon(\omega^{kn}w)$ are nonnegative. Let y_j ($1\le j\le p$) be in $\Gamma_\lambda\cap\Gamma_\lambda^{-1}$ such that $\varepsilon(y_j)=\varepsilon_j(\omega^{kn}w)$. Note that all components of $\varepsilon(y_i)$ are nonnegative and

$y_i \in N$. According to Theorem 6.4.1 and its proof, and Prop. 6.3.7, we have

$$\begin{aligned} t_{w_{n_i}} t_w &= t_{w_{n_i}} t_{\omega^{kn} w} t_{\omega^{-kn} w_\lambda} \\ &= t_{y_1} \cdots t_{y_{i-1}} t_{w_{n_i}} t_{y_i} t_{y_{i+1}} \cdots t_{y_p} t_{\omega^{-kn} w_\lambda} \\ &= t_{y_1} \cdots t_{y_{i-1}} t_{w_{n_i} * y_i} t_{y_{i+1}} \cdots t_{y_p} t_{\omega^{-kn} w_\lambda} \\ &= t_{y_1 * \cdots * y_{i-1} * (w_{n_i} * y_i) * y_{i+1} \cdots * y_p} t_{\omega^{-kn} w_\lambda} \\ &= t_{w_{n_i} * w}. \end{aligned}$$

The lemma is proved.

Lemma 8.1.2. *Let $w \in N$. Recall that $t_{w_j} t_w = \sum_{u \in \Gamma_\lambda \cap \Gamma_\lambda^{-1}} \gamma_{w_j,w,u} t_u$. We have $u \in N$ and $\varepsilon_{ik}(w) \le \varepsilon_{ik}(u) \le \varepsilon_{ik}(w) + 1$ for all k if $\gamma_{w_j,w,u} \neq 0$.*

Proof. First we show that $\varepsilon_{ik}(w) \le \varepsilon_{ik}(u)$. Using Lemma 8.1.1 we see that it is harmless to assume that all components of $\varepsilon(w)$ are non-negative. Using Prop. 6.3.7 we see that u is in N if $\gamma_{w_j,w,u} \neq 0$. Since t_{w_j} appears in $t_{w_1}^j$ with non-zero coefficient, considering $t_{w_1}^j t_w$, by the positivity (see 1.3 (f)) and Prop. 6.3.7 we see that $\varepsilon_{ik}(w) \le \varepsilon_{ik}(u)$ if $\gamma_{w_j,w,u} \neq 0$.

Now we show that $\varepsilon_{ik}(u) \le \varepsilon_{ik}(w)+1$. Let $w_j' = w_{n_i}^{-1} * w_j$. Using Prop. 6.3.8 we see that $t_{w_j'}$ appears in $t_{w_1^{-1}}^{n_i-j}$ with non-zero coefficient. Now consider $t_{w_1^{-1}}^{n_i-j} t_w$. By Lemma 8.1.1 we may assume that all components of $\varepsilon(w)$ are non-positive. By the positivity and Prop. 6.3.8 we see that $\varepsilon_{ik}(u) \le \varepsilon_{ik}(w)$ and u is in N if $\gamma_{w_j',w,u} \neq 0$. Multiplying both sides of $t_{w_j'} t_w = \sum_{u \in N} \gamma_{w_j',w,u} t_u$ by $t_{w_{n_i}}$ and using Lemma 8.1.1 we see that $\varepsilon_{ik}(u) \le \varepsilon_{ik}(w) + 1$ for all k if $\gamma_{w_j,w,u} \neq 0$.

The lemma is proved.

Proposition 8.1.3. *Let $w \in N$. Then $t_{w_j} t_w = \sum_u t_u$, where u runs through the set consisting of all $x \in N$ such that $\varepsilon(x) = \varepsilon(w) + \tau_{ik_1} + \tau_{ik_2} + \cdots + \tau_{ik_j}$ for some sequence $1 \le k_1 < k_2 < \cdots < k_j \le n_i$. See the proof of Theorem 6.4.1 for the definition of τ_{ik}.*

Proof. Using Lemma 8.1.1 we may and will assume that all components of $\varepsilon(w)$ are non-negative.

We use induction on the sum

$$E(w) = \varepsilon_{i1}(w) + \cdots + \varepsilon_{in_i}(w).$$

When $E(w) = 0$, the lemma is trivial. When $E(w) = 1$, the lemma follows from Prop. 6.3.7. Now suppose that the lemma is true when $E(w) \le a$. We need show that the lemma then is true for $E(w) = a + 1$.

We define two bilinear forms, one for J_N, the subring of $J_{\Gamma_\lambda\cap\Gamma_\lambda^{-1}}$ spanned by all t_u $(u \in N)$, the other for $R_{GL_{n_i}(\mathbb{C})}$. We define

$$(\sum a_u t_u, \sum b_u t_u) = \sum a_u b_u$$

and

$$(\sum a_u V_u, \sum b_u V_u) = \sum a_u b_u,$$

where $V_u \in R_{GL_{n_i}(\mathbb{C})}$ stands for an irreducible module of $GL_{n_i}(\mathbb{C})$ of highest weight $u \in N$, recall that we identify N with $\mathbb{Z}^{n_i}_{\mathrm{dom}}$.

Let $u, y \in N$. We say that $t_u \sum_{x\in N} a_x t_x$ ($a_x \in \mathbb{Z}$) has the right y-component if

$$(t_y, t_u \sum_{x\in N} a_x t_x) = (V_y, V_u \sum_{x\in N} a_x V_x).$$

We say that $t_u \sum_{x\in N} a_x t_x$ has the right form if it has the right y-component for all $y \in N$.

By Prop. 6.3.7 and Theorem 7.2.2, we have

(a) if $t_u \sum_{x\in N} a_x t_x$ has the right y-component (resp. right form), then both $(t_u \sum_{x\in N} a_x t_x) t_{w_1}$ and $(t_u \sum_{x\in N} a_x t_x) t_{w_2}$ have the right y-component (resp. right form).

Obviously we have

(b) if $t_u \sum_{x\in N} a_x t_x$ and $t_u \sum b_x t_x$ have the right y-component (resp. right form), then $t_u \sum_{x\in N} (a_x \pm b_x) t_x$ has the right y-component (resp. right form).

We need to show that $t_{w_j} t_w$ has the right from.

Let $\delta_k \in \mathbb{Z}^{n_i}$ be such that its kth component is 1 and other components are 0. By induction hypothesis, we know that $t_{w_j} t_{a\delta_1}$ has the right form, recall that we identify N with $\mathbb{Z}^{n_i}_{\mathrm{dom}}$. Thus

(c) $$t_{w_j} t_{a\delta_1} = t_{(a+1)\delta_1+\delta_2+\cdots+\delta_j} + t_{a\delta_1+\delta_2+\cdots+\delta_j+\delta_{j+1}}.$$

(Convention: If $k > n_i$, then $t_{x+\delta_k} = 0$ for any $x \in \mathbb{Z}^{n_i}_{\mathrm{dom}}$; and if x is in $\mathbb{Z}^{n_i}$ but not in $\mathbb{Z}^{n_i}_{\mathrm{dom}}$, then $t_x = 0$.)

Multiplying t_{w_1} to both sides of (c) and using Prop. 6.3.7, we get

(d) $$\begin{aligned} t_{w_j} t_{a\delta_1} t_{w_1} = {} & t_{w_j}(t_{(a+1)\delta_1} + t_{a\delta_1+\delta_2}) \\ = {} & t_{(a+2)\delta_1+\delta_2+\cdots+\delta_j} + t_{(a+1)\delta_1+2\delta_2+\cdots+\delta_j} + t_{(a+1)\delta_1+\delta_2+\cdots+\delta_{j+1}} \\ & + t_{(a+1)\delta_1+\delta_2+\cdots+\delta_j+\delta_{j+1}} + t_{a\delta_1+2\delta_2+\cdots+\delta_j+\delta_{j+1}} \\ & + t_{a\delta_1+\delta_2+\cdots+\delta_j+\delta_{j+1}+\delta_{j+2}}. \end{aligned}$$

By (d) and Lemma 8.1.2, we see

(e) $t_{(a+2)\delta_1+\delta_2+\cdots+\delta_j}$ appears in $t_{w_j} t_{(a+1)\delta_1}$ with coefficient 1; and all the three elements $t_{(a+1)\delta_1+2\delta_2+\cdots+\delta_j}$, $t_{a\delta_1+2\delta_2+\cdots+\delta_j+\delta_{j+1}}$, $t_{a\delta_1+\delta_2+\cdots+\delta_j+\delta_{j+1}+\delta_{j+2}}$ appear in $t_{w_j} t_{a\delta_1+\delta_2}$ with coefficient 1.

We wish to show that $t_{w_j} t_{(a+1)\delta_1}$ has the right form. By induction hypothesis, $t_{w_j} t_{(a-1)\delta_1+\delta_2}$ has the right form, so $t_{w_j} t_{(a-1)\delta_1+\delta_2} t_{w_1}$ has the right form. We may use Prop. 6.3.7 to expand the expression $t_{w_j} t_{(a-1)\delta_1+\delta_2} t_{w_1}$. Using Lemma 8.1.2 we see that $t_{(a+1)\delta_1+\delta_2+\cdots+\delta_j+\delta_{j+1}}$ appears in $t_{w_j} t_{a\delta_1+\delta_2}$ with coefficient 1. Thus $t_{w_j} t_{a\delta_1+\delta_2}$ has the right form. By (d), $t_{w_j} t_{(a+1)\delta_1}$ has the right form.

We shall use the lexicographical order on $\mathbb{Z}^{n_i}$ with $(1,0,...,0) > (0,1,...,0)$. Now suppose that if all components of w' are non-negative and $E(w') = E(w) = a+1$, $w' > w$, then $t_{w_j} t_{w'}$ has the right form. We need show that $t_{w_j} t_w$ has the right form. Choose k such that $\varepsilon_{ik}(w) > 0$ but $\varepsilon_{il}(w) = 0$ for all $l > k$. If $k = 1$, we have showed that $t_{w_j} t_w$ has the right form. So we may assume that $k > 1$.

By induction hypothesis and (a), $t_{w_j}t_{w-\delta_{k-1}-\delta_k}t_{w_2}$ has the right form. By induction hypothesis again, we see

(f) $t_{w_j}(t_w + t_{w-\delta_{k-1}+\delta_{k+1}} + t_{w-\delta_k+\delta_{k+1}} + t_{w-\delta_{k-1}-\delta_k+\delta_{k+1}+\delta_{k+2}})$ has the right form.

Since $t_{w_j}t_{w-\delta_{k-1}-\delta_k+\delta_{k+1}}t_{w_1}$ has the right form, by induction hypothesis, we get

(g) $t_{w_j}(t_{w-\delta_{k-1}+\delta_{k+1}} + t_{w-\delta_k+\delta_{k+1}} + t_{w-\delta_{k-1}-\delta_k+\delta_{k+1}+\delta_{k+2}})$ has the right form.

Using (b) and (f-g), we see that $t_{w_j}t_w$ has the right form if all components of $\varepsilon(w)$ are nonnegative.

The lemma is proved.

8.2. The based ring $J_{\Gamma_\lambda\cap\Gamma_\lambda^{-1}}$ and the based ring $J_{\mathbf{c}}$

Now we can prove Conjecture 2.3.3.

Theorem 8.2.1 *The map $t_w \to V(\varepsilon(w))$, $w \in \Gamma_\lambda\cap\Gamma_\lambda^{-1}$, defines a ring isomorphism from $J_{\Gamma_\lambda\cap\Gamma_\lambda^{-1}}$ to R_{F_λ}.*

Proof. Use Theorem 6.4.1, Lemma 8.1.3 and 4.2 (f).

Combining Theorem 2.3.2, Theorem 5.2.6 and Theorem 8.2.1, we see that Lusztig Conjecture on based ring for type $\tilde{A}_{n-1}$ is true.

8.2.2. Here we give a summary of our proof of Lusztig Conjecture on based ring for type $\tilde{A}_{n-1}$. Let $\mathbf{c}$ be a two-sided cell of the extended affine Weyl group W associated with $GL_n(\mathbb{C})$. According to Shi [S] and Lusztig [L3], we have a corresponding partition λ of n. Let μ be the dual of λ and $u \in GL_n(\mathbb{C})$ a unipotent element whose Jordan form has partition μ. Then the centralizer of u in $GL_n(\mathbb{C})$ is connected and so is a maximal reductive subgroup F_λ of the centralizer. Thus Lusztig Conjecture on the based ring $J_{\mathbf{c}}$ says that $J_{\mathbf{c}}$ is isomorphic to the $n_\mu \times n_\mu$ matrix algebra over the representation ring R_{F_λ} of F_λ, where n_μ is the number of left cells of W in $\mathbf{c}$.

To prove the required result we first show that $J_{\mathbf{c}}$ is isomorphic to the $n_\mu \times n_\mu$ matrix algebra over the based ring of the intersection of a left cell in $\mathbf{c}$ and its inverse, see Theorem 2.3.2. Then we establish a bijection between the intersection and the set IrrF_λ of isomorphism classes of rational irreducible modules of F_λ in Chapter 5, see Theorem 5.2.6. In Chapters 6-8 we show that the bijection leads to an isomorphism between the based ring of the intersection and R_{F_λ}. This completes our proof of Lusztig Conjecture for type A_{n-1}.

8.2.3. Let u be a unipotent element in $GL_n(\mathbb{C})$ whose Jordan blocks are determined by the dual partition of a partition λ of n. Let $\mathcal{B}_u$ be the variety consisting of all Borel subgroups of $GL_n(\mathbb{C})$ that contain u. In [X4] we show that there is an equivariant F_λ-partition of $\mathcal{B}_u$. Using the partition we determine the equivariant K-group $K^{F_\lambda}(\mathcal{B}_u \times \mathcal{B}_u)$. According to 3.2 (b) in [X4] and Theorems 2.3.2, 5.2.6, 8.2.1, we know that as R_{F_λ}-modules, the equivariant K-group $K^{F_\lambda}(\mathcal{B}_u \times \mathcal{B}_u)$ is isomorphic to $J_{\mathbf{c}}$, where $\mathbf{c}$ is the two-sided cell of W corresponding to λ.

One can define a convolution on $K^{F_\lambda}(\mathcal{B}_u \times \mathcal{B}_u)$. In [L10] Lusztig found some canonical bases for some equivariant K-groups. It is likely that

(1) under the convolution the equivariant K-group $K^{F_\lambda}(\mathcal{B}_u \times \mathcal{B}_u)$ becomes an associative ring with 1,
(2) there exists a canonical $\mathbb{Z}$-basis of $K^{F_\lambda}(\mathcal{B}_u \times \mathcal{B}_u)$ whose elements are one to one corresponding to the elements of the two-sided cell $\mathbf{c}$,
(3) the bijection between the canonical $\mathbb{Z}$-basis in (2) and $\mathbf{c}$ leads to the ring isomorphism between $K^{F_\lambda}(\mathcal{B}_u \times \mathcal{B}_u)$ and $J_{\mathbf{c}}$.
(4) the map from an affine Hecke algebras to the based ring of a two-sided cell of the corresponding extended affine Weyl group defined in [L5] has a natural geometric explanation.

In next section we explain that Lusztig Conjecture on based ring can not be generalized to arbitrary extended affine Weyl groups.

8.3. $PGL_n(\mathbb{C})$

In this section we consider the projective linear group $PGL_n(\mathbb{C})$ of degree n. The extended affine Weyl group associated with $PGL_n(\mathbb{C})$ is just an affine Weyl group of type A_{n-1}. We shall identify it with the affine Weyl group W' in 2.1.3 (c). For each two-sided cell $\mathbf{c}$ (resp. left cell Γ, right cell Φ) of the extended affine Weyl group W associated with $GL_n(\mathbb{C})$, the intersection $\mathbf{c}' = \mathbf{c} \cap W'$ (resp. $\Gamma \cap W'$, $\Phi \cap W'$) is a two-sided cell (resp. left cell, right cell) of W'. Obviously, the based ring $J_{\mathbf{c}'}$ of $\mathbf{c}'$ is a subring of $J_{\mathbf{c}}$.

Now assume that $n = 2$ and $\mathbf{c}'$ is the lowest two-sided cell of W', i.e., the two-sided cell containing s_1, s_0. The two-sided cell $\mathbf{c}'$ contains two left cells. Let Γ_i $(i = 0, 1)$ be the left cell in $\mathbf{c}'$ such that $R(\Gamma_i) = \{s_i\}$. Then

$$\Gamma_1 = \{s_1,\ s_0s_1,\ s_1s_0s_1,\ s_0s_1s_0s_1,\ ...\},$$

$$\Gamma_0 = \{s_0,\ s_1s_0,\ s_0s_1s_0,\ s_1s_0s_1s_0,\ ...\}.$$

The reductive group corresponding to $\mathbf{c}'$ is $F_{\mathbf{c}'} = PGL_2(\mathbb{C})$.

Suppose that there existed a finite $F_{\mathbf{c}'}$-set Y and a bijection $\pi : \mathbf{c}' \to$ the set of isomorphism classes of irreducible $F_{\mathbf{c}'}$ vector bundles on $Y \times Y$ such that the map $t_w \to \pi(w)$ defines a ring isomorphism (preserving the unit element) between $J_{\mathbf{c}'}$ and $K_{F_{\mathbf{c}'}}(Y \times Y)$. Since $F_{\mathbf{c}'}$ is connected, Y must be a trivial $F_{\mathbf{c}'}$-set. Moreover, since $\mathbf{c}'$ contains two left cells, this forces that Y contains two elements. Thus $K_{F_{\mathbf{c}'}}(Y \times Y)$ is isomorphic to the 2×2 matrix algebra $M_2(R_{F_{\mathbf{c}'}})$ over the representation ring $R_{F_{\mathbf{c}'}}$ of $F_{\mathbf{c}'}$. As a consequence there should exist an element w in $\Gamma_0 \cap \Gamma_1^{-1}$ and an element u in $\Gamma_1 \cap \Gamma_0^{-1}$ such that $t_w t_u = t_{s_1}$. But this is impossible since we have

$$\Gamma_0 \cap \Gamma_1^{-1} = \{s_1s_0,\ s_1s_0s_1s_0,\ s_1s_0s_1s_0s_1s_0,\ ...\},$$

$$\Gamma_1 \cap \Gamma_0^{-1} = \{s_0s_1,\ s_0s_1s_0s_1,\ s_0s_1s_0s_1s_0s_1,\ ...\},$$

and

$$t_{(s_1s_0)^k} t_{(s_0s_1)^l} = \sum_{|k-l| \le q \le |k+l-1|} t_{s_1(s_0s_1)^q}.$$

Therefore we can not find the required $F_{\mathbf{c}'}$-set Y and map π for the two-sided cell $\mathbf{c}'$ of W', so that Lusztig Conjecture on based ring can not be generalized to arbitrary extended affine Weyl group. However a weak form of the conjecture might be true which now we are going to state. Let W_G be the extended affine Weyl group associated with a connected reductive algebraic group G over $\mathbb{C}$ and

let $\mathbf{c}$ be a two-sided cell of W_G. Denote by $F_{\mathbf{c}}$ the reductive group corresponding to $\mathbf{c}$. It is likely that for any left cell Γ in $\mathbf{c}$ we can find a finite transitive $F_{\mathbf{c}}$-set Y_Γ such that there exists a bijection $\pi_\Gamma : \Gamma \to$ the set of isomorphism classes of irreducible $F_{\mathbf{c}}$ vector bundles on $Y_\Gamma \times Y_\Gamma$ such that the map $t_w \to \pi(w)$ defines a ring isomorphism (preserving the unit element) between $J_{\Gamma\cap\Gamma^{-1}}$ and $K_{F_{\mathbf{c}'}}(Y_\Gamma \times Y_\Gamma)$.

Now we show that the weak form of Lusztig Conjecture on based ring is true for W'. Let $\mathbf{c}$, Γ_i, Γ_λ, A_{ij} be as in section 2.3. Then $\Gamma'_i = \Gamma_i \cap W'$ is a left cell of W'. Clearly we have

(a) The map $\phi_{ii} : A_{ii} \to A_{11}$ in section 2.3 induces a bijection $\phi'_{ii} : \Gamma'_i \cap {\Gamma'}_i^{-1} \to \Gamma'_1 \cap {\Gamma'}_1^{-1}$.

Using Theorem 2.3.2 (a-b) we get

(b) The map $t_w \to t_{\phi'_{ii}(w)}$ induces a ring isomorphism from $J_{\Gamma'_i\cap\ {\Gamma'}_i^{-1}}$ to $J_{\Gamma'_\lambda\cap{\Gamma'}_\lambda^{-1}}$, where $\Gamma'_\lambda = \Gamma_1 \cap W' = \Gamma_\lambda \cap W'$ is the left cell of W' containing w_λ.

(c) The based ring $J_{\Gamma'_\lambda\cap{\Gamma'}_\lambda^{-1}}$ is commutative.

Recall that we also regard W as a permutation group of $\mathbb{Z}$ (see 2.1). Then W' is a permutation group of $\mathbb{Z}$ consisting of all the permutations σ that satisfy (1) $\sigma(i+n) = \sigma(i) + n$ for all $i \in \mathbb{Z}$ and (2) $\sum_{i=1}^{n}(\sigma(i) - i) = 0$. Thus we have

(d) Let w be in $\Gamma_\lambda \cap \Gamma_\lambda^{-1}$. Then w is in $\Gamma'_\lambda \cap {\Gamma'}_\lambda^{-1}$ if and only if the sum of all $\varepsilon_{ij}(w)$ $(1 \le i \le p,\ 1 \le j \le n_i)$ is 0.

Let $\bar{F}_\lambda$ be the quotient group of F_λ moduloing the center of $GL_n(\mathbb{C})$. Then $\bar{F}_\lambda$ is isomorphic to the reductive group corresponding to the two-sided cell of W' containing w_λ. Let $\mathrm{Dom}(\bar{F}_\lambda)$ be the subset of $\mathrm{Dom}(F_\lambda)$ consisting of all elements (ε_{ij}) in $\mathrm{Dom}(F_\lambda)$ that satisfy $\displaystyle\sum_{\substack{1\le i\le p\\ 1\le j\le n_i}} \varepsilon_{ij} = 0$. Then we have

(e) The map $\varepsilon : \Gamma'_\lambda \cap {\Gamma'}_\lambda^{-1} \to \mathrm{Dom}(\bar{F}_\lambda)$ is a well defined bijection.

(f) The set of isomorphism classes of rational representations of $\bar{F}_\lambda$ is one to one corresponding to the set $\mathrm{Dom}(\bar{F}_\lambda)$.

For $\varepsilon \in \mathrm{Dom}(\bar{F}_\lambda)$ we shall use $V(\varepsilon)$ to denote a rational representation of $\bar{F}_\lambda$ with highest weight ε. According to the above discussion and using Theorem 8.2.1 we get

Theorem 8.3.1. *The map $t_w \to V(\varepsilon(w))$, $w \in \Gamma'_\lambda \cap {\Gamma'}_\lambda^{-1}$, defines a ring isomorphism from $J_{\Gamma'_\lambda\cap{\Gamma'}_\lambda^{-1}}$ to $R_{\bar{F}_\lambda}$.*

In next section we will show that Lusztig Conjecture on based ring is true for the extended affine Weyl group associated with the special linear group $SL_n(\mathbb{C})$.

8.4. $SL_n(\mathbb{C})$

8.4.1. In this section we show that Lusztig Conjecture on based ring is true for the extended affine Weyl group $\bar{W}$ associated with the special linear group $SL_n(\mathbb{C})$ of degree n. It is known that $\bar{W}$ is a quotient group of the extended affine Weyl group W associated with $GL_n(\mathbb{C})$. The kernel is generated by ω^n. For an element w or a subset K in W we shall use $\bar{w}$ or $\bar{K}$ to denote its image in $\bar{W}$. The following property is obvious.

(a) Let w and u be in W. Then w and u are contained in the same left cell (resp. right cell, two-sided cell) of W if and only if $\bar{w}$ and $\bar{u}$ are contained in the same left cell (resp. right cell, two-sided cell) of $\bar{W}$.

Let $\bar{H}$ be the Hecke algebra of $\bar{W}$ over $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$ with parameter q^2. Then $\bar{H}$ is a quotient algebra of the Hecke algebra H of W. The kernel is generated by all $T_{\omega^n w} - T_w$. Let $\bar{w},\ \bar{u},\ \bar{v}$ be in $\bar{W}$. As for W, we can define the Laurant polynomial $h_{\bar{w},\bar{u},\bar{v}}$ by means of the Kazhdan-Lusztig basis of $\bar{H}$ and define the integer $\gamma_{\bar{w},\bar{u},\bar{v}}$ using the Laurant polynomial $h_{\bar{w},\bar{u},\bar{v}}$.

Let w, u, v be in W. Then we have

(b) $h_{w,u,v} = h_{\bar{w},\bar{u},\bar{v}}$.

(c) $\gamma_{w,u,v} = \gamma_{\bar{w},\bar{u},\bar{v}}$.

Let $\bar{J}$ be the based ring of $\bar{W}$ with basis elements $t_{\bar{w}},\ \bar{w} \in \bar{W}$ and structure constants $\gamma_{\bar{w},\bar{u},\bar{v}}$. Then $\bar{J}$ is a quotient ring of the based ring J of W. The kernel is generated by all $t_{\omega^n w} - t_w$. Let $\mathbf{c}$ be a two-sided cell of W and $\bar{\mathbf{c}}$ the two-sided cell of $\bar{W}$ corresponding to $\mathbf{c}$. Then $\bar{J}_{\bar{\mathbf{c}}}$ is a quotient ring of the based ring $J_{\mathbf{c}}$ of $\mathbf{c}$, the kernel is generated by all $t_{\omega^n w} - t_w,\ w \in \mathbf{c}$. Similar conclusion is true for the based ring $J_{\bar{\Gamma}\cap\bar{\Gamma}^{-1}}$, where $\bar{\Gamma}$ is a left cell in $\bar{\mathbf{c}}$.

Let $\Gamma_\lambda,\ \Gamma_i,\ A_{ij}$ and ϕ_{ij} be as in section 2.3. The image in $\bar{W}$ of A_{ij} is denoted by $\bar{A}_{ij}$. Then ϕ_{ij} induces a bijection from $\bar{A}_{ij}$ to $\bar{A}_{11}$. We denote the induced map also by ϕ_{ij}. According to the above discussion and Theorem 2.3.2 we have

Theorem 8.4.2. *Let $\bar{\mathbf{c}}$ be the two-sided cell of $\bar{W}$ corresponding to a partition λ of n and μ the dual partition of λ and $\mathbf{c}$ the corresponding two-sided cell in W.*

(a) *The map $t_{\bar{w}} \to t_{\phi_{ii}(\bar{w})}$ induces a ring isomorphism from $J_{\bar{\Gamma}_i\cap\bar{\Gamma}_i^{-1}}$ to $J_{\bar{\Gamma}_\lambda\cap\bar{\Gamma}_\lambda^{-1}}$.*

(b) *The based ring $J_{\bar{\Gamma}_\lambda\cap\bar{\Gamma}_\lambda^{-1}}$ is commutative.*

(c) *The map $t_{\bar{w}} \to E(t_{\phi_{ij}(\bar{w})}, i, j),\ \bar{w} \in \bar{A}_{ij}$ defines an isomorphism from the based ring $J_{\bar{\mathbf{c}}}$ to $M_{n_\mu}(J_{\bar{\Gamma}_\lambda^{-1}\cap\bar{\Gamma}_\lambda})$, the $n_\mu \times n_\mu$ matrix algebra over the ring $J_{\bar{\Gamma}_\lambda^{-1}\cap\bar{\Gamma}_\lambda}$.*

8.4.3. Let u and F_λ be as in Chapter 4. Then u is in $SL_n(\mathbb{C})$. Let $F'_\lambda = F_\lambda \cap SL_n(\mathbb{C})$. Then F'_λ is a maximal reductive subgroup of the centralizer in $SL_n(\mathbb{C})$ of u. Since F'_λ contains the derived group of F_λ, we have

(a) The restriction to F'_λ of an irreducible rational representation of F_λ is irreducible. Any irreducible representation of F'_λ can be obtained in this way.

Let $\mathrm{Dom}(F_\lambda)$ be as in section 4.1 and r_i be as in section 5.1. The following result is obvious from the definition of F_λ and F'_λ.

(b) Let $\varepsilon = (\varepsilon_{11}, ..., \varepsilon_{pn_p})$ be a dominant weight in $\mathrm{Dom}(F_\lambda)$ and $V(\varepsilon)$ an irreducible representation of F_λ of highest weight ε. Then the restriction to F'_λ of $V(\varepsilon)$ is trivial if and only if there exists an integer k such that $\varepsilon_{ij} = kr_i$ for all $i = 1, 2, ..., p$ and $j = 1, ..., n_i$.

We introduce an equivalence relation on $\mathrm{Dom}(F_\lambda)$. Let $\varepsilon = (\varepsilon_{ij})$ and $\xi = (\xi_{ij})$ be in $\mathrm{Dom}(F_\lambda)$. We say that ε and ξ are equivalent if there exists an integer k such that $\varepsilon_{ij} - \xi_{ij} = kr_i$ for all i, j. The equivalence class containing ε will be denoted by $\bar{\varepsilon}$ and the set of all equivalence classes in $\mathrm{Dom}(F_\lambda)$ is denoted by $\mathrm{Dom}(F'_\lambda)$. By (b) we get

(c) For each element $\bar{\varepsilon}$ in $\mathrm{Dom}(F'_\lambda)$ we have an irreducible rational representation $V(\bar{\varepsilon})$ of F'_λ with highest weight $\bar{\varepsilon}$. The map $\bar{\varepsilon} \to V(\bar{\varepsilon})$ is a bijection from $\mathrm{Dom}(F'_\lambda)$ to the set of isomorphism classes of irreducible rational representations of F'_λ.

Let Γ_λ be as in section 2.3 and let w and u be elements in $\Gamma_\lambda \cap \Gamma_\lambda^{-1}$. It is easy to see that $\bar{w} = \bar{u}$ if and only if $\varepsilon(w)$ and $\varepsilon(u)$ are equivalent. Thus for each $\bar{w}$ in $\bar{\Gamma}_\lambda \cap \bar{\Gamma}_\lambda^{-1}$ we have a well defined element $\varepsilon(\bar{w}) = \overline{\varepsilon(w)}$ in $\mathrm{Dom}(F'_\lambda)$, where w is a preimage in W of $\bar{w}$. Using Theorem 5.2.6 (b) we get

Theorem 8.4.4. *The map $\varepsilon : \bar{w} \to \varepsilon(\bar{w})$ defines a bijection from $\bar{\Gamma}_\lambda \cap \bar{\Gamma}_\lambda^{-1}$ to $Dom(F'_\lambda)$.*

Using 8.4.3 (a), 8.4.1 (c), Theorem 8.2.1 and Theorem 8.4.4 we get

Theorem 8.4.5. *The map $t_{\bar{w}} \to V(\varepsilon(\bar{w}))$ defines a ring isomorphism between the based ring $J_{\bar{\Gamma}_\lambda \cap \bar{\Gamma}_\lambda^{-1}}$ and the representation ring $R_{F'_\lambda}$ of F'_λ.*

Combining Theorems 8.4.2, 8.4.4 and 8.4.5 we see that Lusztig Conjecture on based ring is true for the extended affine Weyl group associated with $SL_n(\mathbb{C})$.

It is expected that the explicit knowledge on the based rings will have applications to understand the representations of Hecke algebras of W. Also we can compute some $\mu(y, w)$ (the coefficient of $q^{\frac{1}{2}(l(w)-l(y)-1)}$ in the Kazhdan-Lusztig polynomial $P_{y,w}$) using the explicit knowledge on the based rings, the details will appear elsewhere.

Bibliography

[G] C. Greene, *Some partitions associated with a partially ordered set,* J. of Combinatorics Theory (A) 20 (1976), 69-79.

[H] J.E. Humphreys, *Introduction to Lie algebras and representation theory,* GTM 9, Springer-Verlag, 1972.

[KL1] D. Kazhdan and G. Lusztig, *Representations of Coxeter groups and Hecke algebras,* Invent. Math. 53 (1979), 165-184.

[KL2] D. Kazhdan and G. Lusztig, *Proof of the Deligne-Langlands Conjecture for Hecke algebras,* Invent. Math. 87 (1987), 153-215.

[IM] N. Iwahori and H. Matsumoto, *On some Bruhat decomposition and the structure of Hecke rings of p-adic Chevalley groups,* Publ. Math. I.H.E.S. 25 (1965) 5-48.

[Li] P. Littelmann, *A Littlewood-Richardson formula for symmetrizable Kac-Moody algebras,* Invent. Math. 116 (1994), 329-346.

[L1] G. Lusztig, *Singularities, character formulas, and a q-analog of weight multiplicities,* Astérisque 101-102 (1983), 208-227.

[L2] G. Lusztig, *Some examples of square integrable representations of semisimple p-adic groups,* Trans. Amer. Math. Soc. 277 (1983), 623-653.

[L3] G. Lusztig, *The two-sided cells of the affine Weyl group of type A_n,* in "Infinite dimensional groups with applications", Springer-Verlag, New York, 1985, pp. 275-287.

[L4] G. Lusztig, *Cells in affine Weyl groups,* in "Algebraic groups and related topics", Advanced Studies in Pure Math., vol. 6, Kinokunia and North Holland, 1985, pp. 255-287.

[L5] G. Lusztig, *Cells in affine Weyl groups, II,* J. Alg. 109 (1987), 536-548.

[L6] G. Lusztig, *Cells in affine Weyl groups, III,* J. Fac. Sci. Univ. Tokyo Sect. IA Math. 34 (1987), 223-243.

[L7] G. Lusztig, *Leading coefficients of character values of Hecke algebras*, Proc. Sympos. Pure Math. vol. 47, part 2, Amer. Math.Soc., Providence, R. I., 1987, 235-262.

[L8] G. Lusztig, *Cells in affine Weyl groups, IV,* J. Fac. Sci. Univ. Tokyo Sect. IA Math. 36 (1989) No. 2, 297-328.

[L9] G. Lusztig, *Periodic W-graphs,* Represent. Theory 1 (1997), 207-279.

[L10] G. Lusztig, *Bases in equivariant K-theory,* preprint, 1998.

[LX] G. Lusztig and N. Xi, *Canonical left cells in affine Weyl groups,* Adv. in Math. 72 (1988), 284-288.

[R1] A. Ram, *Calibrated representations of affine Hecke algebras,* preprint, 1998.

[R2] A. Ram, *Representations of rank two affine Hecke algebras*, preprint, 1998.

[S] J.-Y. Shi, *Kazhdan-Lusztig cells of certain affine Weyl groups,* LNM 1179, Springer-Verlag, Berlin, 1986.

[X1] N. Xi, *The based ring of the lowest two-sided cell of an affine Weyl group,* J. Alg. 134 (1990), 356-368.

[X2] N. Xi, *The based ring of the lowest two-sided cell of an affine Weyl group, II,* Ann. Sci. Ec. Norm. Sup. 27 (1994), 47-61.

[X3] N. Xi, *Representations of affine Hecke algebras,* LNM 1587, Springer-Verlag, Berlin, 1994.

[X4] N. Xi, *A partition of the Springer fibers $\mathcal{B}_N$ for type A_{n-1}, B_2, G_2 and some applications,* Indag. Mathem., N.S., 10 (2) (1994), 307-320.

Index

Notation

§1.1

$\mathcal{A}$

C_w

$l(w)$

$P_{y,w}$

T_w

$y - w$

$y \prec w$

$\mu(y, w)$

§1.2

$a(v)$

$h_{w,u,v}$

$h'_{w,u,v}$

$L(w)$

$R(w)$

$\tilde{T}_w$

$w \underset{L}{\leq} u$, $w \underset{R}{\leq} u$, $w \underset{LR}{\leq} u$

$w \underset{L}{\sim} u$, $w \underset{R}{\sim} u$, $w \underset{LR}{\sim} u$

§1.3

$\mathcal{D}$

P

R

w_0

W_0

X

$\gamma_{w,u,v}$

Ω

§1.4

$D_L(s,t)$, $D_R(s,t)$

${}^*w$, w^*

${}^*w^\star$

$< s, t >$

§1.5

$F_{\mathbf{c}}$

$\mathrm{Irr} F_{\mathbf{c}}$

J, $J_{\mathbf{c}}$, $J_{\Gamma \cap \Gamma^{-1}}$

$R_{F_{\mathbf{c}}}$

t_w

§2.1

s_i
$\tau_1, ..., \tau_i, ..., \tau_n$
ω
Ω

§2.2
n_μ
w_λ
Γ_λ
$\lambda(w)$
$\mu(w)$

§2.3
A_{ij}
F_λ
$M_{n_\mu}(J_{\Gamma_\lambda \cap \Gamma_\lambda^{-1}})$

§3.1
$x_1, ..., x_i, ..., x_n$

§3.2
x_I
x_ν
X^+
$\Gamma_{\mathbf{c}}$
$\Phi_{\mathbf{c}}$

§3.3
m_I
m_x
N

§4.1
$\mathrm{Dom}(F_\lambda)$
F_λ
n_k
p
$\mathbb{Z}^n_{\mathrm{dom}}$

§4.2
$V(x)$

§5.1
a_{ij}
e_i
n_i
p
r_i
Λ_j

§5.2
Z_w
$\varepsilon_{k,i,j}$, $\varepsilon_{k,i,j}(w)$
$\varepsilon(w)$
$\varepsilon_{ij}(w)$
$\varepsilon_{ij}(Z)$

§6.1

$f_{u,v,w}$
W_λ
$\tilde{T}(w)$
§6.2
l_c
m_c
$\varepsilon_i(w)$
§6.4
$x * y$
τ_{il}
§8.1
$w_1, ..., w_i, ..., w_{n_i}$